J. Differential Equations 248 (2010) 693–721

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

# Ramification of rough paths

Massimiliano Gubinelli

*CEREMADE and CNRS (UMR 7534), Université Paris Dauphine, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France*

ARTICLE INFO

*Article history:*
Received 6 March 2008
Revised 4 November 2009
Available online 1 December 2009

*MSC:*
60H99
65L99

*Keywords:*
Rough paths
Rooted trees
Hopf algebras
$B$-series

ABSTRACT

The stack of iterated integrals of a path is embedded in a larger algebraic structure where iterated integrals are indexed by decorated rooted trees and where an extended Chen's multiplicative property involves the Dürr–Connes–Kreimer coproduct on rooted trees. This turns out to be the natural setting for a non-geometric theory of rough paths.



## 1. Introduction

Since the seminal work of Butcher on integration methods [1,2] rooted trees (otherwise called Cayley trees [3]) are recognized as a basic combinatorial structure underlying the numerical and exact solution of ordinary differential equations (see for example [4] and the monograph of Hairer, Nørsett and Wanner [5,4]). Trees are also present in the work of Connes and Kreimer [6–8] on the combinatorial structure of renormalization in perturbative Quantum Field Theory and connections between Runge–Kutta methods and renormalization has been explored by Brouder [9,10]. Connes and Kreimer explored a Hopf algebra structure on rooted trees to disentangle nested sub-divergences in the Feynman diagrams of perturbative QFT. Starting point is the work of Kreimer [11,12] which introduced nested integrals indexed by trees in the analysis of Feynman diagrams. The same Hopf algebra was described before by Dür [13] (for basic results on Hopf algebras see e.g. [14]).

Literature on combinatorial and algebraic properties of rooted trees is quite large, we prefer to single out the work of Hoffman [15] and the two papers of Foissy [16,17] on labeled rooted trees.

*E-mail address:* massimiliano.gubinelli@ceremade.dauphine.fr.

0022-0396/$ – see front matter 
doi:10.1016/j.jde.2009.11.015

A sub-algebra of the Hopf algebra of rooted trees is isomorphic to the Hopf algebra of Chen's iterated integrals [18,19] which is at the base of Lyons theory of rough paths [20]. Lyons theory allows to define and solve differential equations driven by irregular "noises". For an exposition see the work of Lyons cited above, the book of Lyons and Qian [21], the introductory article of Lejay [22]. For alternative approaches to rough paths see the paper [23] of the present author or Feyel and de La Pradelle [24].

Chen [18] showed that a given path in a manifold can be encoded in the Hopf algebra of its iterated integrals. Lyons [20] realized that this encoding is good enough to recover solutions of differential equation driven by such a path.

The aim of the present paper is to build a bridge between rooted trees and rough paths. Here we would like to describe how to encode a control path in a function on labeled rooted trees which we call a *branched rough path* and then generalize the theory of Lyons to build solutions of driven differential equation using this new encoding.

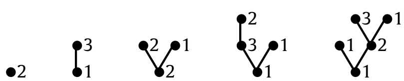
The advantage of this approach is that we can dispose of the notion of *geometric* rough path which is fundamental in Lyons theory. Geometric rough paths possess a rich structure and present nice connections with the geometry of certain Carnot groups [25] but there are situations where the geometric property is not natural, e.g. in Itô stochastic integration or in infinite dimensional generalizations of rough paths [26,27]. A more abstract motivation is to prove that it is possible to build a complete theory of rough paths (at any level of roughness) in the non-geometric setting. Series over trees can be helpful also in the geometric setting: recently Neuenkirch, Nourdin, Rößler and Tindel [28] studied asymptotic expansions for solutions of SDE driven by fractional Brownian motion using expansion over trees.

In Lyons' theory to perform various computations (e.g. Taylor expansions) the geometric condition is (implicitly) used to ensure that products of iterated integrals can be expanded in a sum of other iterated integrals. On the other hand iterated integrals indexed by trees already form a closed algebra with respect to point-wise product and path integration (see below for details). Thus, by enriching the notion of rough path we are able to perform computations as in the case of geometric rough paths and build a complete theory for non-geometric rough integrals. Moreover we hope that such a bridge can inspire novel integration methods for stochastic differential equations in the line of [29].

The plan of the note is the following. In Section 2 we introduce the concept of (labeled) rooted tree, the associated (Dürr–Connes–Kreimer) Hopf algebra and fix the relative notations. In Section 3 we summarize the theory of finite increments described in [23] which can be used as the base for building rough paths theory. In Section 4 we introduce iterated integrals indexed by labeled rooted trees and prove the basic multiplicative property which is a generalization of Chen's multiplicative property for usual iterated integrals. Next, in Section 5 we explain how sums over iterated integrals indexed by rooted trees encode the solutions of driven differential equations. At this point we are ready to generalize rough paths and introduce the notion of *branched rough path* (in Section 7), prove a generalized extension theorem and construct the branched rough path associated to an *almost* branched rough path (following the development of the standard theory, see e.g. [20]). In Section 8, we introduce path controlled by a branched rough path and show how to solve differential equations driven by a branched rough path. Finally in Section 9 we discuss another motivation to consider tree-labeled series: rough paths adapted to the solution of (deterministic or stochastic) infinite dimensional equations.

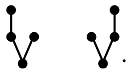
## 2. Trees

Given a finite set $\mathcal{L}$, define an $\mathcal{L}$-labeled rooted tree as a finite graph with a special vertex called *root* such that there is a unique path from the root to any other vertex of the tree. Moreover to each vertex there is associated an element of $\mathcal{L}$. Here some examples of rooted trees labeled by $\mathcal{L} = \{1, 2, 3\}$:

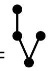


.

We draw the root at the bottom with the tree growing upwards. Note that in a rooted tree the order of the branches at any vertex is ignored so the following two are representations of the same (unlabeled) tree:



Given $k$ $\mathcal{L}$-decorated rooted trees $\tau_1, \ldots, \tau_k$ and a label $a \in \mathcal{L}$ we define $\tau = [\tau_1, \ldots, \tau_k]_a$ as the tree obtained by attaching the $k$ roots of $\tau_1, \ldots, \tau_k$ to a new vertex with label $a$ which will be the root of $\tau$. Any decorated rooted tree can be constructed using the simple decorated tree $\bullet_a$ ($a \in \mathcal{L}$) and the operation $[\cdots]$, e.g.

$[\bullet] =$  $\quad \big[\bullet, [\bullet]\big] =$ , etc.

Denote by $\mathcal{T}_{\mathcal{L}}$ the set of all $\mathcal{L}$ decorated rooted trees and let $\mathcal{T}$ be the set of rooted trees without decoration (i.e. for which the set of labels $\mathcal{L}$ is made of a single element). There is a canonical map $\mathcal{T}_{\mathcal{L}} \to \mathcal{T}$ which simply forget all the labels and every function on $\mathcal{T}$ can be extended, using this map to a function on $\mathcal{T}_{\mathcal{L}}$ for any set of labels $\mathcal{L}$. Let $|\cdot| : \mathcal{T} \to \mathbb{R}$ be the map which counts the number of vertices of the (undecorated) tree and which can be defined recursively as

$$|\bullet| = 1, \qquad \big|[\tau_1, \ldots, \tau_k]\big| = 1 + |\tau_1| + \cdots + |\tau_k|,$$

moreover we define the *tree factorial* $\gamma : \mathcal{T} \to \mathbb{R}$ as

$$\gamma(\bullet) = 1, \qquad \gamma\big([\tau_1, \ldots, \tau_k]\big) = \big|[\tau_1, \ldots, \tau_k]\big| \gamma(\tau_1) \cdots \gamma(\tau_k).$$

Last we define the *symmetry factor* $\sigma : \mathcal{T}_{\mathcal{L}} \to \mathbb{R}$ with the recursive formula $\sigma(\tau) = 1$ for $|\tau| = 1$ and

$$\sigma\big(\big[\tau^1 \cdots \tau^k\big]_a\big) = \frac{k!}{\delta(\tau^1, \ldots, \tau^k)} \sigma\big(\tau^1\big) \cdots \sigma\big(\tau^k\big) \tag{1}$$

where $\delta(\tau^1, \ldots, \tau^k)$ counts the number of different ordered $k$-uples $(\tau^1, \ldots, \tau^k)$ which corresponds to the same (unordered) collection $\{\tau^1, \ldots, \tau^k\}$ of subtrees. The factor $k!/\delta(\tau^1, \ldots, \tau^k)$ counts the order of the subgroup of permutations of $k$ elements which does not change the ordered $k$-uple $(\tau^1, \ldots, \tau^k)$. Then $\sigma(\tau)$ is the order of the subgroup of permutations on the vertex of the tree $\tau$ which do not change the tree (taking into account also the labels). Another equivalent recursive definition for $\sigma$ is

$$\sigma\big(\big[\big(\tau^1\big)^{n_1} \cdots \big(\tau^k\big)^{n_k}\big]_a\big) = n_1! \cdots n_k! \sigma\big(\tau^1\big)^{n_1} \cdots \sigma(\tau_k)^{n_k}$$

where $\tau^1, \ldots, \tau^k$ are distinct subtrees and $n_1, \ldots, n_k$ the respective multiplicities.

Define the algebra $\mathcal{AT}_{\mathcal{L}}$ as the commutative polynomial algebra generated by $\{1\} \cup \mathcal{T}_{\mathcal{L}}$ over $\mathbb{R}$, i.e. elements of $\mathcal{AT}_{\mathcal{L}}$ are finite linear combination with coefficients in $\mathbb{R}$ of formal monomials in the form $\tau_1\tau_2 \cdots \tau_n$ with $\tau_1, \ldots, \tau_n \in \mathcal{T}_{\mathcal{L}}$ or of the unit $1 \in \mathcal{AT}_{\mathcal{L}}$. The set of all tree monomials is the set of *forests* $\mathcal{F}_{\mathcal{L}}$ including the empty forest $1 \in \mathcal{F}_{\mathcal{L}}$. The algebra $\mathcal{AT}_{\mathcal{L}}$ is endowed with a graduation $g$ given by $g(\tau_1 \cdots \tau_n) = |\tau_1| + \cdots + |\tau_n|$ and $g(1) = 0$. This graduation induces a corresponding filtration of $\mathcal{AT}_{\mathcal{L}}$ in finite dimensional linear subspaces $\mathcal{A}_n\mathcal{T}_{\mathcal{L}}$ generated by the set $\mathcal{F}^n_{\mathcal{L}}$ of forests of degree $\leqslant n$.

Any map $f : \mathcal{T}_{\mathcal{L}} \to A$ where $A$ is some commutative algebra, can be extended in a unique way to a homomorphism $f : \mathcal{AT}_{\mathcal{L}} \to A$ by setting: $f(\tau_1 \cdots \tau_n) = f(\tau_1) f(\tau_2) \cdots f(\tau_n)$.

On the algebra $\mathcal{AT}_{\mathcal{L}}$ we can define a counit $\varepsilon : \mathcal{AT}_{\mathcal{L}} \to \mathbb{R}$ as an algebra homomorphism such that $\varepsilon(1) = 1$ and $\varepsilon(\tau) = 0$ otherwise and a *coproduct* $\Delta : \mathcal{AT}_{\mathcal{L}} \to \mathcal{AT}_{\mathcal{L}} \otimes \mathcal{AT}_{\mathcal{L}}$ in the following way: $\Delta$ is an algebra homomorphism, i.e. $\Delta(1) = 1 \otimes 1$, $\Delta(\tau_1 \cdots \tau_n) = \Delta(\tau_1) \cdots \Delta(\tau_n)$ and acts linearly on linear combinations of forests and on each tree it acts recursively as

$$\Delta(\tau) = 1 \otimes \tau + \sum_{a \in \mathcal{L}} \big(B_+^a \otimes \mathrm{id}\big)\big[\Delta\big(B_-^a(\tau)\big)\big] \tag{2}$$

where $B_+^a(1) = \bullet_a$ and $B_+^a(\tau_1 \cdots \tau_n) = [\tau_1 \cdots \tau_n]_a$ and $B_-^a$ is the inverse of $B_+^a$ or is equal to zero if the tree root does not have label $a$, i.e.

$$B_-^a\big(B_+^b(\tau_1 \cdots \tau_n)\big) = \begin{cases} \tau_1 \cdots \tau_n & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

The coproduct $\Delta$ has an explicit description in terms of cuts which is useful in some proofs. A *cut* of a tree $\tau$ is a subset of its edges which is selected to be removed. A cut is *admissible* if going from the root to any leaf of the tree we meet at most one edge belonging to the cut. Given a tree $\tau \in \mathcal{T}_{\mathcal{L}}$ and an admissible cut $c$, we denote with $R_c(\tau) \in \mathcal{T}_{\mathcal{L}}$ the tree obtained after the cut (that is the subgraph containing the root) while the set of subtrees detached from the "trunk" by the cut is denoted by $P_c(\tau) \in \mathcal{F}_{\mathcal{L}}$. With this notation the action of the coproduct on trees $\tau \in \mathcal{T}_{\mathcal{L}}$ can be described by the formula

$$\Delta(\tau) = 1 \otimes \tau + \tau \otimes 1 + \sum_c R_c(\tau) \otimes P_c(\tau) \tag{3}$$

where the sum is performed over all non-trivial admissible cuts $c$ of $\tau$.

Endowed with $\varepsilon$ and $\Delta$ the algebra $\mathcal{AT}_{\mathcal{L}}$ becomes a bialgebra, there exists also an antipode $S$ which completes the definition of the Hopf algebra structure on $\mathcal{T}_{\mathcal{L}}$ as described by Connes and Kreimer [6] (in the unlabeled case).

Note that our definition of the coproduct differs from the one commonly present in the literature by the exchange of the order of the factors in the tensor product in order to be consistent with other notations present in the paper.

There exist various notations for the coproduct $\Delta$, we will often use Sweedler's notation $\Delta\tau = \sum \tau_{(1)} \otimes \tau_{(2)}$ but we also introduce a counting function $c : \mathcal{T}_{\mathcal{L}} \times \mathcal{T}_{\mathcal{L}} \times \mathcal{F}_{\mathcal{L}} \to \mathbb{N}$ such that

$$\Delta\tau = \sum_{\rho \in \mathcal{T}_{\mathcal{L}}, \sigma \in \mathcal{F}_{\mathcal{L}}} c(\tau, \rho, \sigma) \rho \otimes \sigma.$$

In the following we will use letters $\tau, \rho, \sigma, \ldots$ to denote trees in $\mathcal{T}_{\mathcal{L}}$ or forests in $\mathcal{F}_{\mathcal{L}}$, the degree $g(\tau)$ of a forest $\tau \in \mathcal{F}_{\mathcal{L}}$ will also be written as $|\tau|$. Roman letters $a, b, c, \ldots \in \mathcal{L}$ will denote vector indexes (i.e. labels) while $\bar{a}, \bar{b}, \ldots$ will denote multi-indexes with values in $\mathcal{L}$: $\bar{a} = (a_1, \ldots, a_n) \in \mathcal{L}^n$ with $|\bar{a}| = n$ the size of this multi-index.

## 3. Increments

Given $T > 0$, a vector space $V$ and an integer $k \geqslant 1$, we denote by $\mathcal{C}_k(V)$ the set of continuous functions $g : [0, T]^k \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $0 \leqslant i \leqslant k - 1$. Such a function will be called a *k-increment*, and we will set $\mathcal{C}_*(V) = \bigcup_{k \geqslant 1} \mathcal{C}_k(V)$. We write $\mathcal{C}_k = \mathcal{C}_k(\mathbb{R})$. There is a cochain complex $(\mathcal{C}_*(V), \delta)$ where the coboundary $\delta$, satisfying $\delta^2 = 0$, is defined as follows on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \quad (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}, \tag{4}$$

here $\hat{t}_i$ means that this particular argument is omitted. We will denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \operatorname{Ker} \delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \operatorname{Im} \delta$, respectively the spaces of *k-cocycles* and of *k-coboundaries*.

Some simple examples of actions of $\delta$, which will be the ones we will really use throughout the paper, are obtained by letting $g \in \mathcal{C}_1(V)$ and $h \in \mathcal{C}_2(V)$. Then, for any $t, u, s \in [0, T]$, we have $(\delta g)_{ts} = g_t - g_s$, and $(\delta h)_{tus} = h_{ts} - h_{tu} - h_{us}$. Furthermore, it is readily checked [23] that the complex $(\mathcal{C}_*(V), \delta)$ is *acyclic*, i.e. $\mathcal{ZC}_{k+1}(V) = \mathcal{BC}_k(V)$ for any $k \geqslant 1$, or otherwise stated, the sequence

$$0 \to \mathbb{R} \to \mathcal{C}_1(V) \xrightarrow{\delta} \mathcal{C}_2(V) \xrightarrow{\delta} \mathcal{C}_3(V) \xrightarrow{\delta} \mathcal{C}_4(V) \to \cdots \tag{5}$$

is exact. This implies in particular that if $\delta h = 0$ for some $h \in \mathcal{C}_2(V)$ then there exists $f \in \mathcal{C}_1(V)$ such that $\delta f = h$. Thus we get a heuristic interpretation of the coboundary $\delta h$: it measures how much a given 2-increment $h$ is far from being an *exact* increment of a function (i.e. a finite difference).

When $V = \mathbb{R}$ the complex $(\mathcal{C}_*, \delta)$ is an (associative, non-commutative) graded algebra once endowed with the following (exterior) product: for $g \in \mathcal{C}_n$ and $h \in \mathcal{C}_m$ let $gh \in \mathcal{C}_{n+m-1}$ be the element defined by

$$(gh)_{t_1, \ldots, t_{m+n-1}} = g_{t_1, \ldots, t_n} h_{t_n, \ldots, t_{m+n-1}}, \quad t_1, \ldots, t_{m+n-1} \in [0, T]. \tag{6}$$

In this context, the coboundary $\delta$ acts as a graded derivation with respect to the algebra structure. In particular we have the following useful properties.

(i) Let $g, h$ be two elements of $\mathcal{C}_1$. Then

$$\delta(gh) = \delta g\, h + g\, \delta h. \tag{7}$$

(ii) Let $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then

$$\delta(gh) = \delta g\, h + g\, \delta h, \qquad \delta(hg) = \delta h\, g - h\, \delta g.$$

The iterated integrals of smooth functions on $[0, T]$ are particular cases of elements of $\mathcal{C}$ which will be of interest for us. Consider $f \in \mathcal{C}_1^\infty$, where $\mathcal{C}_1^\infty$ is the set of smooth functions from $[0, T]$ to $\mathbb{R}$. For each $h \in \mathcal{C}_2$ the integral $\int_s^t df_u\, h_{us}$, which will be denoted by $\mathcal{J}(df\, h)$, can be considered as an element of $\mathcal{C}_2$. That is, for $s, t \in [0, T]$, we set

$$\mathcal{J}_{ts}(df\, h) = \int_s^t df_u\, h_{us}.$$

The basic relation between integration and the coboundary $\delta$ is given by the next lemma.

**Lemma 3.1.** *Let $h \in \mathcal{C}_2$ such that $\delta h = \sum_i h^{1,i} h^{2,i}$ (finite sum) for $h^{1,i}, h^{2,i} \in \mathcal{C}_2$ and let $x \in \mathcal{C}_1^\infty$. Then*

$$\delta \mathcal{J}(dx\, h) = \mathcal{J}(dx) h + \sum_i \mathcal{J}\big(dx\, h^{(1,i)}\big) h^{(2,i)}. \tag{8}$$

**Proof.**

$$\begin{aligned}\delta\mathcal{J}(dx\,h)_{tus} &= \int_s^t h_{vs}\,dx_v - \int_s^u h_{vs}\,dx_v - \int_u^t h_{vu}\,dx_v \\ &= \int_u^t (h_{vs}-h_{vu})\,dx_v = \int_u^t \delta h_{vus}\,dx_v + \int_u^t h_{us}\,dx_v \\ &= \sum_i \int_u^t h^{(1,i)}_{vu}\,dx_v\,h^{(2,i)}_{us} + \mathcal{J}_{tu}(dx)h_{us}. \qquad \square\end{aligned}$$

Given a vector $\{x^i\}_{i=1,\dots,d}$ of elements of $\mathcal{C}_1^\infty$ introduce iterated integrals recursively as

$$\mathcal{J}\big(dx^{i_1}\,dx^{i_2}\cdots dx^{i_n}\big) = \mathcal{J}\big[dx^{i_1}\,\mathcal{J}\big(dx^{i_2}\cdots dx^{i_n}\big)\big]$$

where $i_1,\dots,i_n \in \{1,\dots,d\}$. Then by using Lemma 3.1 we recover Chen's multiplicative property (in disguise)

$$\delta\mathcal{J}\big(dx^{i_1}\cdots dx^{i_n}\big) = \sum_{k=1}^{n-1} \mathcal{J}\big(dx^{i_1}\cdots dx^{i_k}\big)\mathcal{J}\big(dx^{i_{k+1}}\cdots dx^{i_n}\big), \quad (i_1,\dots,i_n)\in\{1,\dots,d\}^n. \tag{9}$$

## 4. Rooted trees and iterated integrals

Let $x=\{x^a\}_{a=1,\dots,d}$ be a family of smooth elements in $\mathcal{C}_1$ and $\mathcal{L}=\{1,2,\dots,d\}$ the set of indexes. By iterating integrations along the elements of $x$ we can build a map $X:\mathcal{T}_\mathcal{L}\to C([0,T]^2;\mathbb{R})$ defined as follows:

$$X^{\bullet_a}_{ts} = \int_s^t dx^a_u, \qquad X^{[\tau^1\cdots\tau^k]_a}_{ts} = \int_s^t \prod_{i=1}^k X^{\tau^i}_{us}\,dx^a_u. \tag{10}$$

On the vector space $\mathcal{C}_2$ we introduce the associative and commutative inner product $\circ$ as $(a\circ b)_{ts} = a_{ts}b_{ts}$ for $a,b\in\mathcal{C}_2$. With this product $\mathcal{C}_2$ becomes an algebra and as explained before we can extend the map $X:\mathcal{T}_\mathcal{L}\to\mathcal{C}_2$ to a map on $\mathcal{AT}_\mathcal{L}$ by linearity and by letting $X^{\tau_1\cdots\tau_n}_{ts} = X^{\tau_1}_{ts}X^{\tau_2}_{ts}\cdots X^{\tau_n}_{ts}$ for the value of $X$ on the forest $\tau_1\cdots\tau_n$. Using this product we can write $X^{[\tau_1\cdots\tau_n]_a} = \int X^{\tau_1\cdots\tau_n}\,dx^a$.

Let $\mathcal{C}_2^+ = \mathcal{C}_2\oplus e$ be the unital algebra obtained by adding to the algebra $\mathcal{C}_2$ the unit $e$ such that $e_{ts}=1$ for any $t,s\in[0,T]$.

The product $\circ$ has the following relation with $\delta$:

$$\delta(a\circ b) = \delta a\circ\delta b + (ea+ae)\circ\delta b + (eb+be)\circ\delta a + ab + ba \tag{11}$$

where $\circ$ is defined on $\mathcal{C}_3$ in the natural way: $(g\circ h)_{tus} = g_{tus}h_{tus}$ for every $g,h\in\mathcal{C}_3$.

If on the algebra $(\mathcal{C}_2,\circ)$ we consider the exterior product $\mathcal{C}_2\otimes\mathcal{C}_2\to\mathcal{C}_3$ then we can extend the homomorphism $X$ also to the tensor product $\mathcal{AT}_\mathcal{L}\otimes\mathcal{AT}_\mathcal{L}$ by $X^{\sigma\otimes\rho} = X^\sigma X^\rho$ for every $\sigma,\rho\in\mathcal{AT}_\mathcal{L}$.

Denote with $I^a:\mathcal{C}_2\to\mathcal{C}_2$ the integration map given by $I^a(h) = \mathcal{J}(dx^a\,h)$ then for all elements $\sigma\in\mathcal{AT}_\mathcal{L}$ we have $I^aX^\sigma = X^{B^a_+\sigma}$: the map $B^a_+$ represent integration on the sub-algebra $\mathcal{A}_X\subset\mathcal{C}_2^+$ generated by $\{X^\tau\}_{\tau\in\mathcal{T}_\mathcal{L}}$. This sub-algebra contains the polynomial algebra generated by the set $\{\delta x^a\}_{a\in\mathcal{L}}$:

$$X^{\bullet_{a_1}\cdots\bullet_{a_n}} = X^{\bullet_{a_1}}\circ\cdots\circ X^{\bullet_{a_n}} = \delta x^{a_1}\circ\cdots\circ\delta x^{a_n}. \tag{12}$$

It contains also the usual iterated integrals of $x$:

$$\mathcal{J}\big(dx^{a_1}\cdots dx^{a_n}\big) = I^{a_1} I^{a_2}\cdots I^{a_{n-1}}\big(\delta x^{a_n}\big) = X^{B_+^{a_1} B_+^{a_2}\cdots B_+^{a_{n-1}} \bullet_{a_n}} = X^{[\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_1}}. \tag{13}$$

To future use let us denote with $\mathcal{T}_{\mathcal{L}}^{\mathrm{Chen}}$ the subset of $\mathcal{T}_{\mathcal{L}}$ made of "linear" labeled trees of the form $[\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_1}$.

What is remarkable is the relation between the coalgebra structure of the trees and the algebraic properties of the iterated integrals $X$ with respect to the coboundary $\delta$ as illustrated in the next theorem.

**Theorem 4.1** *(Tree multiplicative property). The map $X$ satisfies the following algebraic relation*:

$$\delta X^{\sigma} = X^{\Delta'(\sigma)}, \quad \sigma \in \mathcal{AT}_{\mathcal{L}}, \tag{14}$$

*where $\Delta'$ is the reduced coproduct $\Delta'(\tau) = \Delta(\tau) - 1\otimes\tau - \tau\otimes 1$.*

**Proof.** We will proceed by induction on the degree $g$ of the forests in $\mathcal{AT}_{\mathcal{L}}$ defined above. It is clear that the relation (14) holds for the simple tree $\bullet_a$ with degree $g = 1$. Assume that Eq. (14) holds for every monomial with degree less than $n$ and let us prove it for monomials of degree $n$.

We need the following two properties of the reduced coproduct: first, its recursive definition can be rewritten as

$$\Delta'(\tau) = \sum_{a\in\mathcal{L}} \bullet_a \otimes B_-^a(\tau) + \sum_a \big(B_+^a \otimes \mathrm{id}\big)\big[\Delta'\big(B_-^a(\tau)\big)\big] \tag{15}$$

which follows directly from (2). Next a formula for the action of $\Delta'$ on products of monomials:

$$\Delta'(\rho\sigma) = \Delta'\sigma\,\Delta'\rho + (1\otimes\sigma + \sigma\otimes 1)\Delta'\rho + (1\otimes\rho + \rho\otimes 1)\Delta'\sigma + \rho\otimes\sigma + \sigma\otimes\rho \tag{16}$$

for $\rho, \sigma$ monomials on trees. Assume $g(\rho\sigma) = n$ and let us compute $\delta X^{\rho\sigma}$ using Eq. (11):

$$\begin{aligned}\delta X^{\rho\sigma} &= \delta\big(X^{\rho}\circ X^{\sigma}\big)\\ &= \delta X^{\rho}\circ\big(X^{\sigma}e + eX^{\sigma}\big) + \delta X^{\sigma}\circ\big(X^{\rho}e + eX^{\rho}\big) + \delta X^{\rho}\circ\delta X^{\sigma} + X^{\rho}X^{\sigma} + X^{\sigma}X^{\rho}.\end{aligned}$$

Since $g(\sigma) < n$ and $g(\rho) < n$ we obtain

$$\begin{aligned}\delta X^{\rho\sigma} &= X^{\Delta'\rho}\circ\big(X^{\sigma}e + eX^{\sigma}\big) + X^{\Delta'\sigma}\circ\big(X^{\rho}e + eX^{\rho}\big) + X^{\Delta'\rho}\circ X^{\Delta'\sigma} + X^{\rho}X^{\sigma} + X^{\sigma}X^{\rho}\\ &= X^{\Delta'\rho}\circ X^{\sigma\otimes 1 + 1\otimes\sigma} + X^{\Delta'\sigma}\circ X^{\rho\otimes 1 + 1\otimes\rho} + X^{\Delta'\rho}\circ X^{\Delta'\sigma} + X^{\rho\otimes\sigma} + X^{\sigma\otimes\rho}\\ &= X^{\Delta'\rho(\sigma\otimes 1 + 1\otimes\sigma)} + X^{\Delta'\sigma(\rho\otimes 1 + 1\otimes\rho)} + X^{\Delta'\rho\Delta'\sigma} + X^{\rho\otimes\sigma} + X^{\sigma\otimes\rho}\\ &= X^{\Delta'(\rho\sigma)}\end{aligned}$$

according to Eq. (16). So we have proven Eq. (14) for non-trivial monomials of $g$-degree $n$. It remains to prove the relation for monomials given by a single tree of degree $n$. To do this we need the action of $\delta$ on iterated integrals which is given by Lemma 3.1 above. Let us compute $\delta X^{\tau}$ using formula (8) with $\tau = [\tau_1\cdots\tau_n]_a$:

$$\begin{aligned}\delta X^{[\tau_1\cdots\tau_n]_a} &= \delta\mathcal{J}\big[dx^a\, X^{\tau_1\cdots\tau_n}\big] = \delta x^a\, X^{\tau_1\cdots\tau_n} + \sum_i \mathcal{J}\big[dx^a\, X^{\theta_i^1}\big]\, X^{\theta_i^2}\\ &= X^{\bullet_a} X^{\tau_1\cdots\tau_n} + \sum_i X^{[\theta_i^1]_a}\, X^{\theta_i^2}\end{aligned}$$

where $\delta X^{\tau_1\cdots\tau_n} = \sum_i X^{\theta_i^1} X^{\theta_i^2}$ and $\theta^{1,2}$ satisfy $\Delta'(\tau_1\cdots\tau_n) = \sum_i \theta_i^1 \otimes \theta_i^2$ since our induction assumptions imply that the monomial $\tau_1\cdots\tau_n$, Eq. (14) holds. Then

$$\begin{aligned}\delta X^{[\tau_1\cdots\tau_n]_a} &= X^{\bullet_a\otimes(\tau_1\cdots\tau_n)} + X^{\sum_i[\theta_i^1]_a\otimes\theta_i^2} = X^{\bullet_a\otimes(\tau_1\cdots\tau_n)+\sum_i[\theta_i^1]_a\otimes\theta_i^2}\\ &= X^{\bullet_a\otimes(\tau_1\cdots\tau_n)+(B_+^a\otimes\mathrm{id})(\sum_i\theta_i^1\otimes\theta_i^2)} = X^{\Delta'([\tau_1\cdots\tau_n]_a)},\end{aligned}$$

where we used Eq. (15). Then we proved Eq. (14). $\square$

**Example 4.2.** Let us give an example in one dimension ($d=1$) so trees are not decorated. The forests of degree less or equal to three are:

$$\bullet,\ [\bullet],\ \bullet\bullet,\ [[\bullet]],\ \bullet[\bullet],\ \bullet\bullet\bullet,\ [\bullet\bullet].$$

The reduced coproduct on these monomials acts as follows:

$$\begin{gathered}\Delta'[\bullet] = \bullet\otimes\bullet, \qquad \Delta'(\bullet\bullet) = 2\bullet\otimes\bullet,\\ \Delta'[[\bullet]] = [\bullet]\otimes\bullet + \bullet\otimes[\bullet],\\ \Delta'(\bullet[\bullet]) = \bullet\otimes\bullet\bullet + \bullet\bullet\otimes\bullet + [\bullet]\otimes\bullet + \bullet\otimes[\bullet],\\ \Delta'(\bullet^3) = 3\bullet^2\otimes\bullet + 3\bullet\otimes\bullet^2,\\ \Delta'[\bullet\bullet] = \bullet\otimes\bullet\bullet + 2[\bullet]\otimes\bullet.\end{gathered}$$

So we have

$$\delta X^{[\bullet\bullet]} = X^{\bullet} X^{\bullet\bullet} + 2X^{[\bullet]} X^{\bullet}.$$

**Remark 4.3.** A particular case of the tree multiplicative property (14) is given by Chen's multiplicative property (9) with the aid of the relation (13).

As a first elementary application of this result we derive a tree binomial formula.

**Lemma 4.4** *(Tree binomial). For every $\tau \in \mathcal{T}$ and $a, b \geqslant 0$ we have*

$$(a+b)^{|\tau|} = \sum_i \frac{\tau!}{\tau_i^{(1)}!\tau_i^{(2)}!} a^{|\tau_i^{(1)}|} b^{|\tau_i^{(2)}|}. \tag{17}$$

**Proof.** Consider the iterated integrals $T^\tau$ associated to the identity path $t:\mathbb{R}\to\mathbb{R}$

$$T_{ts}^{\bullet} = t-s, \qquad T_{ts}^{[\tau_1\cdots\tau_n]} = \int_s^t T_{us}^{\tau_1}\cdots T_{us}^{\tau_n}\, du.$$

By induction it is not difficult to prove that $T^\tau_{ts} = (t-s)^{|\tau|}(\tau!)^{-1}$, so applying Theorem 4.1 to $T^\tau$ we get

$$\frac{(t-s)^{|\tau|}}{\tau!} = T^\tau_{ts} = T^\tau_{us} + T^\tau_{tu} + \sum_i{}' T^{\tau_i^{(1)}}_{tu} T^{\tau_i^{(2)}}_{us} = \sum_i T^{\tau_i^{(1)}}_{tu} T^{\tau_i^{(2)}}_{us}$$

$$= \sum_i \frac{1}{\tau_i^{(1)}!\tau_i^{(2)}!}(t-u)^{|\tau_i^{(1)}|}(u-s)^{|\tau_i^{(1)}|}.$$

Then setting $t-u=a$ and $u-s=b$ we get Eq. (17). □

### 4.1. Geometric paths

The above homomorphism $X$ can be simplified using the fact that it is generated by a $C^1$ family $x$. Indeed Chen [18] proved that products of iterated integrals can be always expressed as linear combination of iterated integrals via the *shuffle product*:

$$\mathcal{J}\big(dx^{a_1}\cdots dx^{a_n}\big)\circ\mathcal{J}\big(dx^{b_1}\cdots dx^{b_m}\big) = \sum_{\bar c\in \mathrm{Sh}(\bar a,\bar b)} \mathcal{J}\big(dx^{c_1}\cdots dx^{c_{n+m}}\big) \tag{18}$$

where given two multi-indexes $\bar a = (a_1,\dots,a_n)$ and $\bar b = (b_1,\dots,b_n)$ their *shuffles* $\mathrm{Sh}(\bar a,\bar b)$ is the set of all the possible permutations of the $(n+m)$-uple $(a_1,\dots,a_n,b_1,\dots,b_m)$ which does not change the ordering of the two subsets $\bar a$, $\bar b$.

Using relation (18) we can reduce every $X^\tau$ for $\tau\in\mathcal{T}_\mathcal{L}$ to a linear combination of $\{X^\sigma\}_{\sigma\in\mathcal{T}^{\mathrm{Chen}}_\mathcal{L}}$.

## 5. Series solutions of driven differential equations

Under appropriate conditions on the vectorfield $f:\mathbb{R}^n\to\mathbb{R}^n$ the solution $y$ of the differential equation $dy/dt = f(y)$ $y_0=\eta$ admits the series representation

$$y_t = \eta + \sum_{\tau\in\mathcal{T}} \psi^f(\tau)(\eta)\frac{t^{|\tau|}}{\sigma(\tau)\tau!} \tag{19}$$

which is called $B$-series (in honor of J. Butcher, see [1,4,5]). The coefficients $\psi^f$ are called *elementary differentials* and are defined as

$$\psi^f(\bullet)(\xi) = f(\xi),\qquad \psi^f\big(\big[\tau^1\cdots\tau^k\big]\big) = \sum_{\bar b\in\mathcal{IL}_1} f_{\bar b}(\eta)\psi^f\big(\tau^1\big)(\xi)^{b_1}\cdots\psi^f\big(\tau^k\big)(\xi)^{b_k}$$

where we introduce multi-indexes $\bar b\in\mathcal{IL}_1 = \bigcup_{k=0}^\infty \mathcal{L}_1^k$, $\mathcal{L}_1=\{1,\dots,n\}$, with the convention $\mathcal{L}_1^0=\emptyset$ and we set $f_\emptyset(\xi)=f(\xi)$ and $f_{\bar b}(\xi) = \prod_{i=1}^{|\bar b|}\partial_{\xi_{b_i}} f(\xi)$ for the derivatives of the vectorfield.

In this section we study the analogous series expansion for *driven* differential equation. Consider a $C^1$ path $x:[0,T]\to\mathbb{R}^d$ and let $\{x^a\}_{a\in\mathcal{L}}$ be its coordinates in a fixed basis. Fix a point $\eta\in\mathbb{R}^n$ and let $f_a:\mathbb{R}^n\to\mathbb{R}^n$, $a=1,\dots,d$, be a collection of analytic vectorfields on $\mathbb{R}^n$. Let $R$ be a common analiticy radius around $\eta$ for all coordinates.

**Theorem 5.1.** *The solution of the differential equation $dy_t = \sum_{a\in\mathcal{L}} f_a(y_t)\,dx^a_t$, $y_0 = \eta$ admit locally the series representation*

$$\delta y_{ts} = \sum_{\tau\in\mathcal{T}_{\mathcal{L}}} \frac{1}{\sigma(\tau)} \phi^f(\tau)(y_s) X^\tau_{ts}, \qquad y_0 = \eta \tag{20}$$

*where the sum runs over all $\mathcal{L}$-labeled rooted trees $\tau \in \mathcal{T}_{\mathcal{L}}$ and where we recursively define functions $\phi^f : \mathcal{T}_{\mathcal{L}} \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\phi^f(\bullet_a)(\xi) = f_a(\xi), \qquad \phi^f\big(\big[\tau^1\cdots\tau^k\big]_a\big)(\xi) = \sum_{\bar b\in\mathcal{IL}_1:\ |\bar b|=k} f_{a;b_1\cdots b_k}(\xi)\prod_{i=1}^k \big[\phi^f\big(\tau^i\big)(\xi)\big]^{b_i}.$$

**Proof.** Let us assume for the moment that the series (20) converges absolutely. We will verify that Eq. (20) satisfies the integral equation

$$\delta y_{ts} = \sum_{a\in\mathcal{L}} \int_s^t f_a(y_u)\,dx^a_u. \tag{21}$$

Consider the Taylor series for $f$ around $\xi \in \mathbb{R}^n$:

$$f_a\big(\xi'\big) = \sum_{\bar b\in\mathcal{IL}_1} \frac{f_{a;\bar b}(\xi)}{|\bar b|!} \prod_{i=1}^{|\bar b|} \big(\xi' - \xi\big)^{b_i}$$

where $\xi^k$ is the $k$-th coordinate of the vector $\xi \in \mathbb{R}^n$. By the analyticity of the vectorfields $f_a$ this series converges as long as $|\xi - \xi'| \leqslant R - |\xi' - \eta|$.

Compute the r.h.s. of Eq. (21) by plugging in Eq. (20) and the Taylor expansion of $f$:

$$\begin{aligned}
&\sum_{a\in\mathcal{L}} \int_s^t f_a(y_u)\,dx^a_u \\
&\quad= \sum_{a\in\mathcal{L}} \sum_{\bar b\in\mathcal{IL}_1} \frac{f_{a;\bar b}(y_s)}{|\bar b|!} \int_s^t \left( \prod_{i=1}^{|\bar b|} \delta y^{b_i}_{us} \right) dx^a_u \\
&\quad= \sum_{a\in\mathcal{L}} \sum_{\bar b\in\mathcal{IL}_1} \frac{f_{a;\bar b}(y_s)}{|\bar b|!} \int_s^t \prod_{i=1}^{|\bar b|} \left[ \sum_{\tau\in\mathcal{T}_{\mathcal{L}}} \frac{1}{\sigma(\tau)} \big[\phi^f(\tau)(y_s)\big]^{b_i} X^\tau_{us} \right] dx^a_u \\
&\quad= \sum_{a\in\mathcal{L}} \sum_{\bar b\in\mathcal{IL}_1} \frac{f_{a;\bar b}(y_s)}{|\bar b|!} \sum_{\tau^1,\dots,\tau^{|\bar b|}} \frac{1}{\sigma(\tau^1)\cdots\sigma(\tau^{|\bar b|})} \left( \prod_{i=1}^{|\bar b|} \big[\phi^f\big(\tau^i\big)(y_s)\big]^{b_i} \right) \int_s^t \prod_{i=1}^{|\bar b|} X^{\tau^i}_{us}\,dx^a_u \\
&\quad= \sum_{a\in\mathcal{L}} \sum_{k=0}^\infty \sum_{\tau^1,\dots,\tau^k} \frac{1}{k!\sigma(\tau^1)\cdots\sigma(\tau^k)} \sum_{\bar b\in\mathcal{IL}_1:\ |\bar b|=k} f_{a;\bar b}(y_s) \left( \prod_{i=1}^k \big[\phi^f(\tau)(y_s)\big]^{b_i} \right) \int_s^t \prod_{i=1}^k X^{\tau^i}_{us}\,dx^a_u
\end{aligned}$$

$$= \sum_{a\in\mathcal{L}} \sum_{k=0}^{\infty} \sum_{\tau^1,\ldots,\tau^k} \frac{1}{\sigma([\tau^1\cdots\tau^k]_a)\delta(\tau^1,\ldots,\tau^k)} \phi^f\big(\big[\tau^1\cdots\tau^k\big]_a\big)(y_s) X_{ts}^{[\tau^1\cdots\tau^k]_a}$$
$$= \sum_{\tau\in\mathcal{T}_{\mathcal{L}}} \frac{1}{\sigma(\tau)} \phi^f(\tau)(y_s) X_{ts}^{\tau}$$

which proves the claim. Note the multiplicity factor $\delta$ which disappears from the last line.

To prove the absolute convergence of the series we need bounds on $X^\tau$ and $\phi^f(\tau)$. For $X^\tau$ we have:

$$\big|X_{ts}^\tau\big| \leqslant \frac{[A|t-s|]^{|\tau|}}{\tau!}$$

where $A = \sup_{t\in[0,T]} |\dot{x}_t|$. This bound can be easily proven inductively on $\tau$.

Since $f_a$ are analytic functions, from Cauchy inequalities we obtain

$$\big|f_{a,\bar{b}}(y_s)\big| \leqslant \theta(\bar{b}) M (R-r_s)^{-|\bar{b}|} \leqslant |\bar{b}|! M (R-r_s)^{-|\bar{b}|} \leqslant g^{(|\bar{b}|)}(r_s),$$

see e.g. [5, p. 47], where $r_s = |y_s - \eta|$ and $M$ is a constant depending only on $\{f_a\}_{a\in\mathcal{L}}$ and where we introduced the function $g(r) = MR(R-r)^{-1}$ and its derivatives $g^{(k)}(r) = MRk!(R-r)^{-k-1}$. Define "elementary differentials" $\psi : \mathcal{T} \times [0,R) \to \mathbb{R}$ for $g$ as

$$\psi(\bullet)(r) = g(r), \qquad \psi\big([\tau_1\cdots\tau_k]\big)(r) = g^{(k)}(r) k! M (R-r)^{-k}.$$

Then we have the bounds $|\phi^f(\tau)(y_s)| \leqslant \psi(\tau)(r_s)$ for any $\tau \in \mathcal{T}_{\mathcal{L}}$ and the series (20) can be bounded by

$$\sum_{\tau\in\mathcal{T}_{\mathcal{L}}} \frac{1}{\sigma(\tau)} \psi(\tau)(r_s) A^{|\tau|} \frac{|t-s|^{|\tau|}}{\tau!}$$

and by taking into account the multiplicity $d^{|\tau|}$ of labeled trees corresponding to the same tree $\tau$ we get

$$\sum_{\tau\in\mathcal{T}} \frac{1}{\sigma(\tau)} \psi(\tau)(r_s) (dA)^{|\tau|} \frac{|t-s|^{|\tau|}}{\tau!}.$$

This series is exactly the $B$-series (19) for the solution $r_t$ of the differential equation

$$\frac{dr_t}{dt} = dA\, g(r_t) = dA\, MR(R-r_t)^{-1}, \qquad r_0 = 0 \tag{22}$$

when written starting from $r_s$ at time $s < t$. Then

$$r_t = r_s + \sum_{\tau\in\mathcal{T}} \frac{1}{\sigma(\tau)} \psi(\tau)(r_s) (dA)^{|\tau|} \frac{|t-s|^{|\tau|}}{\tau!}$$

as long as the solution $r_t$ exists and has a power series expansion in $t-s$. But the explicit solution of Eq. (22) is given by $r_t = R(1-\sqrt{1-t/t_*})$ with $t_* = R/(2dA\,M)$ and has power series expansion for any $t < t_*$. So the original series is summable at least for any $t, s \in [0, t_*)$. $\square$

In the rest of this section we will denote $y_s^\tau = \phi^f(\tau)(y_s)/\sigma(\tau)$ so that $\delta y_{ts} = \sum_{\tau\in\mathcal{T}_\mathcal{L}} X^\tau_{ts} y^\tau_s$, moreover we will use the convention $X^\emptyset_{ts} = 1$ and $y^\emptyset_s = y_s$ to write

$$y_t = \sum_{\tau\in\mathcal{T}_\mathcal{L}\cup\{\emptyset\}} X^\tau_{ts} y^\tau_s.$$

The recursion for $y^\tau$ reads

$$y_s^{\bullet_a} = f_a(y_s), \qquad y_s^{[\tau^1\cdots\tau^k]_a} = \frac{\sigma(\tau^1)\cdots\sigma(\tau^k)}{\sigma(\tau)} \sum_{\bar b:\ |\bar b|=k} f_{a,\bar b}(y_s) y_s^{\tau_1,b_1}\cdots y_s^{\tau_k,b_k}. \tag{23}$$

We have the following theorem which shows that each of the paths $y^\tau$ can be expanded in series w.r.t. $X$ with coefficients which depends on the combinatorics of the reduced coproduct:

**Theorem 5.2.** *For any $\tau\in\mathcal{T}_\mathcal{L}\cup\{\emptyset\}$ we have*

$$\delta y^\tau_{ts} = \sum_{\sigma\in\mathcal{T}_\mathcal{L},\ \rho\in\mathcal{F}_\mathcal{L}} c'(\sigma,\tau,\rho) X^\rho_{ts} y^\sigma_s \tag{24}$$

*where $c'$ is the counting function for the reduced coproduct: $\Delta'\sigma = \sum_{\tau,\rho} c'(\sigma,\tau,\rho)\tau\otimes\rho$.*

**Proof.** The proof is by induction on $\tau$. The case $\tau = \bullet_a$ requires only Taylor expansion:

$$\begin{aligned}
\delta y^{\bullet_a}_{ts} &= \delta f_a(y)_{ts} = \sum_{\bar b} \frac{f_{a;\bar b}(y)}{|\bar b|!} (\delta y_{ts})^{\bar b} \\
&= \sum_{k\geqslant 1}\sum_{\tau^1,\dots,\tau^k}\sum_{\bar b:\ |\bar b|=k} \frac{f_{a;\bar b}(y)}{k!} y_s^{\tau^1,b_1}\cdots y_s^{\tau^k,b_k} X_{ts}^{\tau^1\cdots\tau^k} \\
&= \sum_{k\geqslant 1}\sum_{\tau^1,\dots,\tau^k} \frac{\sigma([\tau^1\cdots\tau^k]_a)}{k!\sigma(\tau^1)\cdots\sigma(\tau^k)} y_s^{[\tau^1\cdots\tau^k]_a} X_{ts}^{\tau^1\cdots\tau^k} \\
&= \sum_{k\geqslant 1}\sum_{\tau^1,\dots,\tau^k} \frac{1}{\delta(\tau^1,\dots,\tau^k)} y_s^{[\tau^1\cdots\tau^k]_a} X_{ts}^{\tau^1\cdots\tau^k} \\
&= \sum_\tau c'(\tau,\bullet_a,\rho) y^\tau_s X^\rho_{ts}
\end{aligned} \tag{25}$$

since $c'(\tau,\bullet_a,\rho)$ is different from zero, and take value one, iff $\tau = [\rho]_a$.

Now, assume Eq. (24) holds for all $\tau\in\mathcal{T}^n_\mathcal{L}$ and let us prove that it holds for trees $\tau$ with $|\tau| = n+1$. So take $\tau = [\tau^1\cdots\tau^k]_a$ with $|\tau| = n+1$, then $|\tau^i|\leqslant n$ for any $i = 1,\dots,k$. To compute the action of the map $\delta$ on $y^\tau$ we use the recursive relation (23):

$$\delta y_{ts}^{[\tau^1\cdots\tau^k]_a} = \frac{\sigma(\tau^1)\cdots\sigma(\tau^k)}{\sigma(\tau)} \sum_{\bar b:\ |\bar b|=k} \delta\big[f_{a,\bar b}(y) y^{\tau^1,b_1}\cdots y^{\tau^k,b_k}\big]_{ts} \tag{26}$$

and the Leibniz formula

$$\delta\big(g^1\cdots g^k\big)_{ts} = \big(g^1_s + \delta g^1_{ts}\big)\cdots\big(g^1_s + \delta g^1_{ts}\big) - g^1_s\cdots g^k_s = \sum_G G^1_{ts}\cdots G^k_{ts}$$

where the sum is over all possible choices of $G$-s such that $G^i_{ts} = g^i_s$ or $G^i_{ts} = \delta g^i_{ts}$ excluding the case where all the $G$-s are $g$ (that is, there should be at least one factor of the form $\delta g^i$). By Taylor expansion

$$\delta f_{a,\bar b}(y)_{ts} = \sum_{m\geqslant 1}\sum_{\bar c:\ |\bar c|=m}\frac{f_{a,\bar b\bar c}(y)_s}{m!}\sum_{\eta^1,\dots,\eta^m} y^{\eta^1,c_1}_s\cdots y^{\eta^m,c_m}_s X^{\eta^1\cdots\eta^m}_{ts}$$

while using the induction hypothesis we have

$$\delta y^{\tau^i} = \sum_{\rho^i,\zeta^i} c\big(\zeta^i,\tau^i,\rho^i\big)X^{\rho^i}y^{\zeta^i} = \sum_{\zeta^i} X^{\zeta^i_{(2)}}y^{\zeta^i}\delta_{\tau^i,\zeta^i_{(1)}}$$

where there is an implicit sum over the terms $\zeta^i_{(1)}, \zeta^i_{(2)}$ in the reduced coproduct of $\zeta^i$ and where $\delta_{\tau^i,\zeta^i_{(1)}}$ denotes the Kronecker delta function. Then we rewrite Eq. (26) as

$$\begin{aligned}\delta y^{[\tau^1\cdots\tau^k]_a}_{ts} = {} & \frac{\sigma(\tau^1)\cdots\sigma(\tau^k)}{\sigma(\tau)} \\ & \times\sum_{m\geqslant 0}\frac{1}{m!}\sum_{\zeta^1,\dots,\zeta^k}\sum_{\eta^1,\dots,\eta^m}\sum_{\bar c:\ |\bar c|=m+k} f_{a,\bar c}(y_s)y^{\eta^1,c_1}_s\cdots y^{\eta^m,c_m}_s y^{\zeta^1,c_{m+1}}_s\cdots y^{\zeta^k,c_{m+k}}_s \\ & \times X^{\eta^1\cdots\eta^m\cdots\zeta^1_{(2)}\cdots\zeta^k_{(2)}}_{ts}\delta_{\tau^1,\zeta^1_{(1)}}\cdots\delta_{\tau^k,\zeta^k_{(1)}}.\end{aligned}\tag{27}$$

The summation in this formula has to be understood as follows: the sum over $\zeta^i$ is performed on all trees which contains $\tau^i$ in the sense that $c'(\zeta^i,\tau^i,\rho^i)$ is different form zero for some $\rho^i$ and on the tree $\zeta^i = \tau^i$ in which case we understand that $\zeta^i_{(1)} = \tau^i$ and $\zeta^i_{(2)} = \emptyset$ (the empty forest). Note that this case in not contained in the reduced coproduct but is generated by the Leibniz's formula. Moreover we implicitly exclude from the summation above the case when $m = 0$ and all the $\zeta^i$ are equal to the corresponding $\tau^i$. Then with this proviso we can simplify the above formula as

$$\begin{aligned}\delta y^{[\tau^1\cdots\tau^k]_a}_{ts} = {} & \frac{\sigma(\tau^1)\cdots\sigma(\tau^k)}{\sigma(\tau)}\sum_{m\geqslant 0}\frac{1}{m!}\sum_{\zeta^1,\dots,\zeta^k}\sum_{\eta^1,\dots,\eta^m}\frac{\sigma(\zeta)}{\sigma(\zeta^1)\cdots\sigma(\zeta^k)\sigma(\eta^1)\cdots\sigma(\eta^m)} \\ & \times X^{\eta^1\cdots\eta^m\cdots\zeta^1_{(2)}\cdots\zeta^k_{(2)}}y^\zeta\delta_{\tau^1,\zeta^1_{(1)}}\cdots\delta_{\tau^k,\zeta^k_{(1)}}\end{aligned}\tag{28}$$

where $\zeta = [\zeta^1\cdots\zeta^k\eta^1\cdots\eta^k]$. Now, recalling Eq. (1), write

$$\begin{aligned}\delta y^{[\tau^1\cdots\tau^k]_a}_{ts} = {} & \sum_{m\geqslant 0}\sum_{\zeta^1,\dots,\zeta^k}\sum_{\eta^1,\dots,\eta^m}\frac{(k+m)!}{k!m!}\frac{\delta(\tau^1,\dots,\tau^k)}{\delta(\zeta^1,\dots,\zeta^k,\eta^1,\dots,\eta^k)} \\ & \times X^{\eta^1\cdots\eta^m\cdots\zeta^1_{(2)}\cdots\zeta^k_{(2)}}y^\zeta\delta_{\tau^1,\zeta^1_{(1)}}\cdots\delta_{\tau^k,\zeta^k_{(1)}}.\end{aligned}\tag{29}$$

Introduce a new function $\tilde c : \mathcal{T}_{\mathcal{L}} \times \mathcal{T}_{\mathcal{L}} \times \mathcal{F}_{\mathcal{L}} \to \mathbb{N}$ such that

$$\tilde c(\kappa_1,\kappa_2,\kappa_3) = \begin{cases} c'(\kappa_1,\kappa_2,\kappa_3) & \text{for } \kappa_3 \neq \emptyset, \\ \delta_{\kappa_1,\kappa_2} & \text{for } \kappa_3 = \emptyset \end{cases}$$

which counts the number of ways to cut away a forest $\kappa_3$ from the tree $\kappa_1$ leaving the tree $\kappa_2$ where we allow the empty cut which leaves the tree intact. Using $\tilde c$ we rewrite the last equation as

$$\delta y_{ts}^{[\tau^1\cdots\tau^k]_a} = \sum_{m\geqslant 0} \sum_{\zeta^1,\ldots,\zeta^{k+m}} \sum_{\theta^1,\ldots,\theta^m} \frac{(k+m)!}{k!m!} \frac{\delta(\tau^1,\ldots,\tau^k)}{\delta(\zeta^1,\ldots,\zeta^{k+m})} \tilde c\big(\zeta^1,\tau^1,\theta^1\big)\cdots \tilde c\big(\zeta^k,\tau^k,\theta^k\big) \times y^\zeta X^{\zeta^1\cdots\zeta^m\theta^1\cdots\theta^k} \tag{30}$$

where now $\zeta = [\zeta^1\cdots\zeta^{k+m}]_a$ and $\zeta^1,\ldots,\zeta^{k+m} \in \mathcal{T}_{\mathcal{L}}$ are non-empty trees and $\theta^1,\ldots,\theta^k \in \mathcal{F}_{\mathcal{L}}$ are possibly empty forests but we exclude the case when $m=0$ and all the $\theta^i$ are empty. Now we will show that this expression corresponds exactly to

$$\delta y_{ts}^{[\tau^1\cdots\tau^k]_a} = \sum_{m\geqslant 0} \sum_{\zeta\in\mathcal{T}_{\mathcal{L}}:\ \zeta=[\zeta^1\cdots\zeta^{k+m}]_a} c'\big(\zeta,\big[\tau^1\cdots\tau^k\big]_a,\theta\big) X^\theta y^\zeta \tag{31}$$

which is what we want to prove. Note that the restriction in the sum over trees $\zeta$ of the form $[\zeta^1\cdots\zeta^{k+m}]_a$ for some $m\geqslant 0$ is due to the fact that for trees with less than $k$ branches at the origin the factor $c(\zeta,\tau,\theta)$ is zero.

Each forest $\zeta^1\cdots\zeta^{k+m}$ appears $\delta(\zeta^1,\ldots,\zeta^{k+m})$ times in the summation, moreover given the tree $\zeta=[\zeta^1\cdots\zeta^{k+m}]_a$ there are $(k+m)!/(k!m!)$ ways to choose $m$ branches of the root to cut away. Let us say that these cuts are on the last $m$ branches $\zeta^{k+1},\ldots,\zeta^{k+m}$. Then the rest of the cuts appears on the first $k$ and for a fixed set $\zeta^1,\ldots,\zeta^k$ of trees to cut there are $\delta(\tau^1,\ldots,\tau^k)$ possible ways of associating each $\tau$ to some $\zeta$ to determine the associated cuts (if they are possible at all). Chosen the pairing between the $\zeta$-s and the $\tau$-s there are $\prod_{i=1}^k \sum_{\theta^i\in\mathcal{F}_{\mathcal{L}}} \tilde c(\zeta^i,\tau^i,\theta^i)$ possible cuts (note that chosen $\zeta^i$ and $\tau^i$ the forest $\theta^i$ is uniquely determined). Moreover since either $m>0$ or some $\theta^i\neq\emptyset$ there is at least one proper (i.e. not empty nor full) cut in Eq. (30). This concludes the proof. $\square$

## 6. Integration of finite increments

We recall the integration theory introduced in [23] in some details since this setting is quite different from the original rough path theory developed in [21,20].

Notice that our future discussions will mainly rely on $k$-increments with $k\leqslant 3$. We measure the size of these increments by Hölder-like norms: for $f\in\mathcal{C}_2(V)$ let

$$\|f\|_\mu = \sup_{s,t\in[0,T]} \frac{|f_{ts}|}{|t-s|^\mu} \quad \text{and} \quad \mathcal{C}_1^\mu(V) = \big\{f\in\mathcal{C}_2(V);\ \|f\|_\mu<\infty\big\}.$$

In the same way, for $h\in\mathcal{C}_3(V)$, set

$$\|h\|_{\gamma,\rho} = \sup_{s,u,t\in[0,T]} \frac{|h_{tus}|}{|u-s|^\gamma|t-u|^\rho},$$
$$\|h\|_\mu = \inf\bigg\{\sum_i \|h_i\|_{\rho_i,\mu-\rho_i};\ h=\sum_i h_i,\ 0<\rho_i<\mu\bigg\}, \tag{32}$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0, z)$. We set

$$\mathcal{C}_3^\mu(V) = \{h \in \mathcal{C}_3(V);\ \|h\|_\mu < \infty\}.$$

Eventually, let $\mathcal{C}_3^{1+}(V) = \bigcup_{\mu>1} \mathcal{C}_3^\mu(V)$, and remark that the same kind of norms can be considered on the spaces $\mathcal{Z}\mathcal{C}_3(V)$, leading to the definition of the spaces $\mathcal{Z}\mathcal{C}_3^\mu(V)$ and $\mathcal{Z}\mathcal{C}_3^{1+}(V)$.

With these notations in mind, the following proposition is a basic result which is at the core of our approach to path-wise integration:

**Proposition 6.1** *(The $\Lambda$-map). There exists a unique linear map $\Lambda : \mathcal{Z}\mathcal{C}_3^{1+}(V) \to \mathcal{C}_2^{1+}(V)$ such that*

$$\delta\Lambda = \mathrm{Id}_{\mathcal{Z}\mathcal{C}_3(V)}.$$

*Furthermore, for any $\mu > 1$, this map is continuous from $\mathcal{Z}\mathcal{C}_3^\mu(V)$ to $\mathcal{C}_2^\mu(V)$ and we have*

$$\|\Lambda h\|_\mu \leqslant \frac{1}{2^\mu - 2}\|h\|_\mu, \quad h \in \mathcal{Z}\mathcal{C}_3^{1+}(V). \tag{33}$$

We can now give an algorithm for a canonical decomposition of the preimage of $\mathcal{Z}\mathcal{C}_3^{1+}(V)$, or in other words, of a function $g \in \mathcal{C}_2(V)$ whose increment $\delta g$ is small enough:

**Corollary 6.2.** *Take an element $g \in \mathcal{C}_2(V)$ such that $\delta g \in \mathcal{C}_3^\mu(V)$ for $\mu > 1$. Then $g$ can be decomposed in a unique way as $g = \delta f + \Lambda\delta g$, where $f \in \mathcal{C}_1(V)$.*

*For any 2-increment $g \in \mathcal{C}_2(V)$, such that $\delta g \in \mathcal{C}_3^{1+}(V)$, set $\delta f = (\mathrm{Id} - \Lambda\delta)g$. Then*

$$(\delta f)_{ts} = \lim_{|\Pi_{ts}|\to 0} \sum_{i=0}^{n} g_{t_{i+1}t_i},$$

*where the limit is over any partition $\Pi_{ts} = \{t_0 = t, \dots, t_n = s\}$ of $[t, s]$ whose size tends to zero.*

**Proof.** See [23]. □

## 7. Branched rough paths

Up to this point we have considered only properties of the iterated integrals of smooth functions $\{x^a\}_{a\in\mathcal{L}}$ however from the algebraic point of view the only data we need to build the family $\{X^\tau\}_{\tau\in\mathcal{T}_\mathcal{L}}$ is a family of maps $\{I^a : \mathcal{C}_2 \to \mathcal{C}_2\}_{a\in\mathcal{L}}$ satisfying certain properties.

**Definition 7.1.** We call *integral* a linear map $I : \mathcal{D}_I \to \mathcal{D}_I$ on a sub-algebra $\mathcal{D}_I \subset \mathcal{C}_2^+$ satisfying two properties:

$$I(hf) = I(h)f, \quad \forall h \in \mathcal{D}_I,\ f \in \mathcal{C}_1,$$

and

$$\delta I(h) = I(e)h + \sum_i I(h^{1,i})h^{2,i}, \quad \text{when } h \in \mathcal{D}_I,\ \delta h = \sum_i h^{1,i}h^{2,i} \text{ and } h^{1,i} \in \mathcal{D}_I.$$

We explicitly require that $e \in \mathcal{D}_I$.

Using the embedding $f \in \mathcal{C}_1 \mapsto fe \in \mathcal{C}_2$ we can extend the map $I$ to $\mathcal{C}_1$: for any $f \in \mathcal{C}_1$ we let $I(f) = I(fe)$ and since $fe = ef + \delta f$ (as easily verified) we have

$$I(f) = I(e)f + I(\delta f)$$

for any $f \in \mathcal{C}_1$ such that $\delta f \in \mathcal{D}_I$.

Given a family $\{I^a\}_{a \in \mathcal{L}}$ of such integral maps on a common algebra $\mathcal{D} \subseteq \mathcal{C}_2$ we can associate to them a family $\{X^\tau\}_{\tau \in \mathcal{F}_\mathcal{L}}$ recursively as done in Section 4 above:

$$X^{\bullet_a} = I^a(e), \qquad X^{[\tau^1 \cdots \tau^k]_a} = I^a\big(X^{\tau^1 \cdots \tau^k}\big), \qquad X^{\tau^1 \cdots \tau^k} = X^{\tau^1} \circ \cdots \circ X^{\tau^k}.$$

In this way we establish an algebra homomorphism from $\mathcal{AT}_\mathcal{L}$ to a subalgebra of $\mathcal{C}_2$ generated by the $X^\tau$-s. This homomorphism sends the operation $B_+^a$ on $\mathcal{AT}_\mathcal{L}$ to the integral map $I^a$ on $\mathcal{C}_2$. It is not difficult to verify that Theorem 4.1 extends to the map $X$ generated by the family $\{I^a\}_a$.

Let us now introduce a regularity condition on the map $X$. Given $\gamma \in (0,1]$ define the function $q_\gamma$ on forests as $q_\gamma(\tau) = 1$ for $|\tau| \leqslant 1/\gamma$ and

$$q_\gamma(\tau) = \frac{1}{2^{\gamma|\tau|} - 2} {\sum}' q_\gamma\big(\tau^{(1)}\big) q_\gamma\big(\tau^{(2)}\big) \tag{34}$$

whenever $\tau \in \mathcal{T}$ with $|\tau| > 1/\gamma$ and $q_\gamma(\tau_1 \cdots \tau_n) = q_\gamma(\tau_1) \cdots q_\gamma(\tau_n)$ for $\tau_1, \ldots, \tau_n \in \mathcal{T}$.

Note that $q_\gamma$ satisfies also the equation

$$q_\gamma(\tau) = \frac{1}{2^{\gamma|\tau|}} \sum q_\gamma\big(\tau^{(1)}\big) q_\gamma\big(\tau^{(2)}\big)$$

which involves the splitting given by the coproduct $\Delta$ while the definition (34) involves the splitting of trees given by the reduced coproduct $\Delta'$.

**Definition 7.2.** We call a homomorphism $X : \mathcal{AT} \to \mathcal{C}_2$ a *branched rough path* (BRP) of roughness $\gamma > 0$, if it satisfies Eq. (14) and moreover is such that

$$\big\|X^\tau\big\|_{\gamma|\tau|} \leqslant BA^{|\tau|} q_\gamma(\tau), \quad \tau \in \mathcal{F}_\mathcal{L}, \tag{35}$$

for some constants $B \in [0,1]$ and $A \geqslant 0$.

Under certain conditions we can extend a homomorphism $X : \mathcal{A}_n\mathcal{T} \to \mathcal{C}_2$ defined only on the sub-algebra of trees with degree less or equal to $n$ to the whole algebra.

**Theorem 7.3.** *Let us give a partial homomorphism $X : \mathcal{A}_n\mathcal{T}_\mathcal{L} \to \mathcal{C}_2$ satisfying Eq.* (14) *and such that there exist positive constants $\gamma$, $A \geqslant 0$, $B \in [0,1]$ for which*

$$\big\|X^\tau\big\|_{\gamma|\tau|} \leqslant BA^{|\tau|} q_\gamma(\tau), \quad \tau \in \mathcal{T}_\mathcal{L}^n, \tag{36}$$

*with $\gamma(n+1) > 1$. Then there exists a unique extension of $X$ to a branched rough path defined on the whole $\mathcal{AT}$ with roughness $\gamma$ and such that Eq.* (36) *holds for any $\tau \in \mathcal{T}_\mathcal{L}$.*

**Proof.** We proceed by induction and assume that we have already found an extension $X : \mathcal{A}_m \mathcal{T}_{\mathcal{L}} \to \mathcal{C}_2$ satisfying Eq. (14) and for which we have

$$\left\| X^\tau \right\|_{\gamma|\tau|} \leqslant B q_\gamma(\tau) A^{|\tau|}, \quad \tau \in \mathcal{T}_{\mathcal{L}}^m. \tag{37}$$

This is true if $m = n$. Let us prove that we can extend $X$ to the set of trees with degree $m+1$ with the same bound on the Hölder norms. Since $\gamma m \geqslant \gamma(n+1) > 1$ we can set $X^\tau := \Lambda[X^{\Delta'\tau}]$ for every $\tau$ such that $|\tau| = m$. Indeed

$$\left\| X^{\Delta'\tau} \right\|_{m\gamma} \leqslant \sum_i \left\| X^{\tau_i^{(1)} \otimes \tau_i^{(2)}} \right\|_{m\gamma} \leqslant \sum_i{}' \left\| X^{\tau_i^{(1)}} \right\|_{|\tau_i^{(1)}|\gamma} \left\| X^{\tau_i^{(2)}} \right\|_{|\tau_i^{(2)}|\gamma}$$

since $|\tau_i^{(1)}| + |\tau_i^{(2)}| = m$ for every $i$. This shows that $X^{\Delta'\tau} \in \mathcal{C}_2^{m\gamma}$ and so it is in the domain of $\Lambda$ if $\delta X^{\Delta'\tau} = 0$, but by the induction hypothesis

$$\begin{aligned} \delta X^{\Delta'\tau} &= \delta \sum_i{}' X^{\tau_i^{(1)}} X^{\tau_i^{(2)}} = \sum_i{}' \big[\delta X^{\tau_i^{(1)}}\big] X^{\tau_i^{(2)}} - \sum_i{}' X^{\tau_i^{(1)}} \big[\delta X^{\tau_i^{(2)}}\big] \\ &= X^{(\mathrm{id}\otimes\Delta')\Delta'\tau - (\Delta'\otimes\mathrm{id})\Delta'\tau} = 0 \end{aligned}$$

by coassociativity of the reduced coproduct. To prove the bound on $X^\tau$ recall that

$$\begin{aligned} \left\| X^\tau \right\|_{\gamma|\tau|} = \left\| \Lambda X^{\Delta'\tau} \right\|_{\gamma|\tau|} &\leqslant \frac{1}{2^{|\tau|\gamma} - 2} \sum_i{}' \left\| X^{\tau_i^{(1)}} \right\|_{|\tau_i^{(1)}|\gamma} \left\| X^{\tau_i^{(2)}} \right\|_{|\tau_i^{(2)}|\gamma} \\ &\leqslant B^2 \frac{1}{2^{|\tau|\gamma} - 2} \sum_i{}' A^{|\tau_i^{(1)}| + |\tau_i^{(2)}|} q_\gamma\big(\tau_i^{(1)}\big) q_\gamma\big(\tau_i^{(2)}\big) \\ &\leqslant B^2 A^{|\tau|} q_\gamma(\tau) \end{aligned}$$

and since $B \leqslant 1$ we have the required bound. □

**Remark 7.4.** While we have not been able to prove any asymptotic behavior for $q_\gamma(\tau)$ as $|\tau| \to \infty$ we conjecture that

$$q_\gamma(\tau) \asymp C_1 C_2^{|\tau|} (\tau!)^{-\gamma} \tag{38}$$

for some constants $C_1$ and $C_2$. For the class of linear Chen trees $\mathcal{T}^{\mathrm{Chen}}$ this conjecture is true thanks to the inequality

$$\sum_{k=0}^n \frac{a^{\gamma k} b^{\gamma(n-k)}}{(k!)^\gamma (n-k!)^\gamma} \leqslant c_\gamma \frac{(a+b)^{\gamma n}}{(n!)^\gamma} \tag{39}$$

valid for any $\gamma \in (0,1]$ and $a, b \geqslant 0$ and where the constant $c_\gamma$ depends only on $\gamma$. We prove this inequality in Appendix A. Note that this inequality is a variant of Lyons' neo-classical inequality (see e.g. [20]) which in our notations reads

$$\sum_{k=0}^n \frac{a^{\gamma k} b^{\gamma(n-k)}}{(\gamma k)! [\gamma(n-k)n]!} \leqslant c_\gamma \frac{(a+b)^{\gamma n}}{(\gamma n)!}. \tag{40}$$

A sufficient condition for the validity of the conjecture would be the existence of a "neo-classical tree inequality" of the form

$$\sum \frac{a^{\gamma|\tau^{(1)}|}b^{\gamma|\tau^{(2)}|}}{(\tau^{(1)}!)^{\gamma}(\tau^{(2)}!)^{\gamma}} \leqslant c_{\gamma}\frac{(a+b)^{\gamma|\tau|}}{(\tau!)^{\gamma}} \tag{41}$$

for any $\tau \in \mathcal{T}$. The inequality is true when $\gamma = 1$ by using the tree binomial formula given in Lemma 4.4.

The asymptotic behavior (38) appears also in the estimation of tree-indexed iterated integrals in the context of 3d Navier–Stokes equation studied in [30] (see also Section 9).

We denote with $\Omega^{\gamma}_{\mathcal{T},\mathcal{L}}$ the space of $\gamma$-BRP, on this space we can introduce a distance by letting

$$d_{\gamma}(X,Y) = \sum_{\tau\in\mathcal{F}^n_{\mathcal{L}}} \|X^{\tau} - Y^{\tau}\|_{\gamma|\tau|}$$

where $n$ is again the largest integer such that $n\gamma \leqslant 1$. This distance is strong enough to separate points in $\Omega^{\gamma}_{\mathcal{T},\mathcal{L}}$:

**Corollary 7.5.** *If $X, Y \in \Omega^{\gamma}_{\mathcal{T},\mathcal{L}}$ and $d_{\gamma}(X,Y) = 0$ then $X = Y$.*

**Proof.** If we let $Z^{\tau} = X^{\tau} - Y^{\tau}$ for $\tau \in \mathcal{T}^n_{\mathcal{L}}$ then the partial homomorphism $Z$ is such that $Z^{\tau} = 0$ and satisfy Eq. (14) for all $\tau \in \mathcal{T}^n_{\mathcal{L}}$. Then we can choose $B = 0$ and an arbitrary $A$ in the bounds (36) and use Theorem 7.3 to conclude that we must have $Z^{\tau} = 0$ for any $\tau \in \mathcal{T}_{\mathcal{L}}$, i.e. that $X = Y$. $\square$

**Definition 7.6.** An *almost branched rough path* (aBRP) is a partial homomorphism $\widetilde{X} : \mathcal{A}_n\mathcal{T} \to \mathcal{C}_2$ such that it approximately satisfies Eq. (14) for any tree $\tau \in \mathcal{T}^n_{\mathcal{L}}$ modulus an element of $\mathcal{C}^{1+}_3$ and for which we have

$$\max_{\tau\in\mathcal{T}^n_{\mathcal{L}}} \|\widetilde{X}^{\tau}\|_{\gamma|\tau|} \leqslant K \tag{42}$$

for some constant $K$ and some $\gamma > 1/(n+1)$.

Then we have the following result:

**Theorem 7.7.** *For any aBRP $\widetilde{X}$ there is a unique BRP $X$ of roughness $\gamma$ such that*

$$\max_{\tau\in\mathcal{T}^n_{\mathcal{L}}} \|X^{\tau} - \widetilde{X}^{\tau}\|_{(n+1)\gamma} < \infty.$$

**Proof.** The assumption is that $\delta\widetilde{X}^{\tau} = \widetilde{X}^{\Delta'\tau} + R^{\tau}$ where $R^{\tau} \in \mathcal{C}^{(n+1)\gamma}_3$ for any $\tau \in \mathcal{F}^n_{\mathcal{L}}$.

We will set $X^{\tau} = \widetilde{X}^{\tau} + Q^{\tau}$ and determine the increments $Q^{\tau}$ by induction. First look at $\tau$ such that $|\tau| = 1$, in this case

$$\delta X^{\tau} = \delta\widetilde{X}^{\tau} + \delta Q^{\tau} = R^{\tau} + \delta Q^{\tau}$$

since $\Delta'\tau = 0$. Then we set $Q^{\tau} = -\Lambda R^{\tau}$ since $R^{\tau} \in \mathcal{Z}\mathcal{C}^{1+}_3$. So that we obtain $\delta X^{\tau} = 0$ as it should be. Now assume that for $\tau \in \mathcal{T}^m_{\mathcal{L}}$ we have obtained $Q^{\tau}$ such that $\delta X^{\tau} = X^{\Delta'\tau}$ and let us find such corrections $Q^\tau$ for $\tau \in \mathcal{T}_{\mathcal{L}}^{m+1}$ with $|\tau| = m+1$. We have

$$\delta \widetilde{X}^\tau = {\sum}' \widetilde{X}^{\tau^{(1)}} \widetilde{X}^{\tau^{(2)}} + R^\tau$$

since both $\tau^{(1)}$ and $\tau^{(2)}$ have degree less than $m+1$ we can apply the induction hypothesis and obtain $\delta \widetilde{X}^\tau = \sum' (X^{\tau^{(1)}} - Q^{\tau^{(1)}})(X^{\tau^{(2)}} - Q^{\tau^{(2)}}) + R^\tau$. Now let

$$\widetilde{R}^\tau = {\sum}' \big[ Q^{\tau^{(1)}} X^{\tau^{(2)}} + X^{\tau^{(1)}} Q^{\tau^{(2)}} - Q^{\tau^{(1)}} Q^{\tau^{(2)}} \big] - R^\tau$$

so that $\delta \widetilde{X}^\tau - \widetilde{R}^\tau = \sum' X^{\tau^{(1)}} X^{\tau^{(2)}}$. If we can show that $\widetilde{R}^\tau \in \mathcal{Z}\mathcal{C}_3^{1+}$, then setting $Q^\tau = \Lambda[\widetilde{R}^\tau]$ we would have obtained $\delta X^\tau = \delta \widetilde{X}^\tau - \widetilde{R}^\tau = \sum' X^{\tau^{(1)}} X^{\tau^{(2)}}$ as required and the induction would be complete. It is clear that $\widetilde{R}^\tau \in \mathcal{C}_3^{1+}$. The only problem is to prove that it is in the image of $\delta$. By the triviality of the complex $(\mathcal{C}_*, \delta)$ this is equivalent to show that $\delta \widetilde{R}^\tau = 0$. Note that

$$\delta \widetilde{R}^\tau = \delta \left[ \delta \widetilde{X}^\tau - {\sum}' X^{\tau^{(1)}} X^{\tau^{(2)}} \right] = -\delta {\sum}' X^{\tau^{(1)}} X^{\tau^{(2)}}.$$

Using again the induction hypothesis we get

$$\begin{aligned} \delta \widetilde{R}^\tau &= {\sum}' X^{\tau^{(1)}} \delta X^{\tau^{(2)}} - {\sum}' \delta X^{\tau^{(1)}} X^{\tau^{(2)}} \\ &= {\sum}' X^{\tau^{(1)}} X^{\Delta' \tau^{(2)}} - {\sum}' X^{\Delta' \tau^{(1)}} X^{\tau^{(2)}} = X^{(\mathrm{id} \otimes \Delta')\Delta' \tau} - X^{(\Delta' \otimes \mathrm{id})\Delta' \tau}. \end{aligned}$$

But now $\delta \widetilde{R}^\tau = X^{(\mathrm{id} \otimes \Delta')\Delta' \tau - (\Delta' \otimes \mathrm{id})\Delta' \tau} = 0$ since the reduced coproduct is coassociative. The proof of uniqueness is left to the reader. □

## 8. Controlled paths

Following the line of development of [23] we describe now a sufficiently large class of paths which can be integrated against a given $\gamma$-branched rough path $X$. We then show that this set of paths constitutes an algebra and that integration and application of sufficiently regular maps preserve this class. It will constitute the natural space where to look for solutions of rough differential equations driven by a branched path.

In Section 5 we showed that the solution $y$ of a driven differential equation has the form of a series indexed by trees: $\delta y_{ts} = \sum_{\tau \in \mathcal{T}_{\mathcal{L}}} X_{ts}^\tau y_s^\tau$ (cf. Eq. (20)) for suitable coefficients functions $\{y^\tau\colon \tau \in \mathcal{T}_{\mathcal{L}}\}$ which satisfy Eq. (24).

This suggests the following:

**Definition 8.1.** Let $X$ be a $\gamma$-BRP and let $n$ be the largest integer such that $n\gamma \leqslant 1$. For any $\kappa \in (1/(n+1), \gamma]$ a path $y$ is a *$\kappa$-weakly controlled* by $X$ with values in $V$ if there exist paths $\{y^\tau \in \mathcal{C}_2^{|\tau|\kappa}(V)\colon \tau \in \mathcal{F}_{\mathcal{L}}^{n-1}\}$ and remainders $\{y^\sharp \in \mathcal{C}_2^{n\kappa}(V), y^{\sharp,\tau} \in \mathcal{C}_2^{(n-|\tau|)\kappa}(V), \tau \in \mathcal{F}_{\mathcal{L}}^{n-1}\}$ such that

$$\delta y = \sum_{\tau \in \mathcal{F}_{\mathcal{L}}^{n-1}} X^\tau y^\tau + y^\sharp \tag{43}$$

and for $\tau \in \mathcal{F}_{\mathcal{L}}^{n-1}$:

$$\delta y^{\tau} = \sum_{\sigma \in \mathcal{F}_{\mathcal{L}}^{n-1}} \sum_{\rho} c'(\sigma, \tau, \rho) X^{\rho} y^{\sigma} + y^{\tau,\sharp} \tag{44}$$

where we mean $\delta y^{\tau} = y^{\tau,\sharp}$ when $|\tau| = n - 1$. We denote $\mathcal{Q}_{\kappa}(X; V)$ the vector space of $\kappa$-weakly controlled paths by $X$ with values in $V$. Fixed a norm $|\cdot|$ on $V$ we introduce a norm $\|\cdot\|_{\mathcal{Q},\kappa}$ on $\mathcal{Q}_{\kappa}(X; V)$ as

$$\|y\|_{\mathcal{Q},\kappa} = |y_0| + \|y^{\sharp}\|_{n\kappa} + \sum_{\tau \in \mathcal{F}_{\mathcal{L}}^{n-1}} \|y^{\tau,\sharp}\|_{\kappa(n-|\tau|)}.$$

To be precise, a well-defined element in $\mathcal{Q}_{\kappa}(X; V)$ is given by specifying the path $y$ and all its "derivatives" $\{y^{\tau}\}_{\tau}$ but we usually omit this for the sake of brevity. A path in $\mathcal{Q}_{\kappa}(X; V)$ has a partial expansion in $X$ with a remainder denoted with $y^{\sharp}$. Likewise every coefficient path in this expansion has a similar expansion of progressively lower order. We write $\mathcal{Q}_{\kappa}(X) = \mathcal{Q}_{\kappa}(X; \mathbb{R})$.

**Example 8.2.** Let us give an example for $d = 1$ of the structure of a controlled path (since $d = 1$ the partial series are indexed by unlabeled trees). Take $\gamma > 1/5$ so that $n = 4$ and assume that $X$ is a $\gamma$-BRP. Then $y \in \mathcal{Q}_{\gamma}$ corresponds to the set of paths

$$y \in \mathcal{C}_1^{\gamma}, \qquad y^{\bullet} \in \mathcal{C}_1^{\gamma}, \qquad y^{[\bullet]}, y^{\bullet\bullet} \in \mathcal{C}_1^{2\gamma}, \qquad y^{[\bullet\bullet]}, y^{[\bullet]\bullet} y^{[\bullet\bullet]}, y^{[[\bullet]]}, y^{\bullet\bullet\bullet} \in \mathcal{C}_1^{3\gamma}$$

satisfying the following algebraic relations

$$\begin{aligned}
\delta y &= X^{\bullet} y^{\bullet} + X^{[\bullet]} y^{[\bullet]} + X^{\bullet\bullet} y^{\bullet\bullet} + X^{[\bullet\bullet]} y^{[\bullet\bullet]} + X^{[\bullet]\bullet} y^{[\bullet]\bullet} + X^{[\bullet\bullet]} y^{[\bullet\bullet]} + X^{\bullet\bullet\bullet} y^{\bullet\bullet\bullet} + X^{[[\bullet]]} y^{[[\bullet]]} + y^{\sharp},\\
\delta y^{\bullet} &= X^{\bullet}\big(y^{[\bullet]} + 2y^{\bullet\bullet}\big) + X^{[\bullet]}\big(y^{[[\bullet]]} + y^{[\bullet]\bullet}\big) + X^{\bullet\bullet}\big(y^{[\bullet]\bullet} + y^{[\bullet\bullet]} + 3y^{\bullet\bullet\bullet}\big) + y^{\bullet,\sharp},\\
\delta y^{[\bullet]} &= X^{\bullet}\big(y^{[\bullet]\bullet} + 2y^{[\bullet\bullet]} + y^{[[\bullet]]}\big) + y^{[\bullet],\sharp},\\
\delta y^{\bullet\bullet} &= X^{\bullet}\big(y^{[\bullet]\bullet} + y^{\bullet\bullet\bullet}\big) + y^{\bullet\bullet,\sharp},\\
\delta y^{[\bullet\bullet]} &= y^{[\bullet\bullet],\sharp},\\
\delta y^{[\bullet]\bullet} &= y^{[\bullet]\bullet,\sharp},\\
\delta y^{\bullet\bullet\bullet} &= y^{\bullet\bullet\bullet,\sharp},\\
\delta y^{[[\bullet]]} &= y^{[[\bullet]],\sharp}
\end{aligned}$$

with remainders of orders

$$y^{\sharp} \in \mathcal{C}_2^{4\gamma}, \quad y^{\bullet,\sharp} \in \mathcal{C}_2^{3\gamma}, \qquad y^{[\bullet],\sharp}, y^{\bullet\bullet,\sharp} \in \mathcal{C}_2^{2\gamma}, \qquad y^{[\bullet\bullet],\sharp}, y^{[\bullet]\bullet,\sharp}, y^{\bullet\bullet\bullet,\sharp}, y^{[[\bullet]],\sharp} \in \mathcal{C}_2^{\gamma}.$$

The following lemma will be useful in computations below.

**Lemma 8.3.**

$$\delta y^{\sharp} = \sum_{\tau \in \mathcal{F}_{\mathcal{L}}^{n-1}} X^{\tau} y^{\tau,\sharp}.$$

**Proof.**

$$
\begin{aligned}
\delta y^{\sharp} &= \sum_{\tau\in\mathcal{F}_{\mathcal{L}}^{n-1}} X^{\tau}\delta y^{\tau} - \sum_{\tau\in\mathcal{F}_{\mathcal{L}}^{n-1}} \delta X^{\tau} y^{\tau} \\
&= \sum_{|\tau|=n-1} X^{\tau}\delta y^{\tau} + \sum_{\tau\in\mathcal{F}_{\mathcal{L}}^{n-2}} X^{\tau}\Big(\sum_{\sigma\in\mathcal{F}_{\mathcal{L}}^{n-1}}\sum_{\rho} c'(\sigma,\tau,\rho)X^{\rho}y^{\sigma} + y^{\tau,\sharp}\Big) - \sum_{\sigma\in\mathcal{F}_{\mathcal{L}}^{n-1}} \delta X^{\sigma} y^{\sigma} \\
&= \sum_{|\tau|=n-1} X^{\tau}\delta y^{\tau} + \sum_{\tau\in\mathcal{F}_{\mathcal{L}}^{n-2}}\sum_{\sigma\in\mathcal{F}_{\mathcal{L}}^{n-1}}\sum_{\rho} c'(\sigma,\tau,\rho)X^{\tau}X^{\rho}y^{\sigma} + \sum_{\tau\in\mathcal{F}_{\mathcal{L}}^{n-2}} X^{\tau}y^{\tau,\sharp} - \sum_{\sigma\in\mathcal{F}_{\mathcal{L}}^{n-1}} \delta X^{\sigma} y^{\sigma} \\
&= \sum_{|\tau|=n-1} X^{\tau}\delta y^{\tau} + \sum_{\sigma\in\mathcal{F}_{\mathcal{L}}^{n-1}}\sum_{\tau\in\mathcal{F}_{\mathcal{L}}^{n-2},\rho} c'(\sigma,\tau,\rho)X^{\tau}X^{\rho}y^{\sigma} + \sum_{\tau\in\mathcal{F}_{\mathcal{L}}^{n-2}} X^{\tau}y^{\tau,\sharp} - \sum_{\sigma\in\mathcal{F}_{\mathcal{L}}^{n-1}} \delta X^{\sigma} y^{\sigma} \\
&= \sum_{|\tau|=n-1} X^{\tau}\delta y^{\tau} + \sum_{\sigma\in\mathcal{F}_{\mathcal{L}}^{n-1}} \big(X^{\Delta'\sigma} - \delta X^{\sigma}\big)y^{\sigma} + \sum_{\tau\in\mathcal{F}_{\mathcal{L}}^{n-2}} X^{\tau}y^{\tau,\sharp} \\
&= \sum_{\tau\in\mathcal{F}_{\mathcal{L}}^{n-1}} X^{\tau}y^{\tau,\sharp}. \qquad \square
\end{aligned}
$$

**Lemma 8.4.** *Let $\varphi \in C_b^n(\mathbb{R}^k,\mathbb{R})$ and $y \in \mathcal{Q}_\kappa(X;\mathbb{R}^k)$, then $z_t = \varphi(y_t)$ is a weakly controlled path, $z \in \mathcal{Q}_\kappa(X;\mathbb{R})$, and its coefficients are given by*

$$
z^{\tau} = \sum_{m=1}^{n-1}\sum_{\substack{\bar b\in\mathcal{I}\mathcal{L}_1\\ |\bar b|=m}} \frac{\varphi_{\bar b}(y)}{m!} \sum_{\substack{\tau_1,\dots,\tau_m\in\mathcal{F}_{\mathcal{L}}^{n-1}\\ \tau_1\cdots\tau_m=\tau}} y^{\tau_1,b_1}\cdots y^{\tau_m,b_m}, \quad \tau\in\mathcal{F}_{\mathcal{L}}^{n-1},
$$

*where $\mathcal{L}_1 = \{1,\dots,k\}$ (note that all the summations are over a finite number of terms).*

**Proof.** The Taylor expansion for $\varphi$ reads

$$
\varphi(\xi') = \varphi(\xi) + \sum_{m=1}^{n-1}\sum_{\substack{\bar b\in\mathcal{I}\mathcal{L}_1\\ |\bar b|=m}} \frac{\varphi_{\bar b}(\xi)}{m!}\big(\xi'-\xi\big)^{\bar b} + O\big(|\xi'-\xi|^{n}\big)
$$

which plugged into $\delta z = \delta\varphi(y)$ gives

$$
\begin{aligned}
\delta z_{ts} &= \sum_{m=1}^{n-1}\sum_{\substack{\bar b\in\mathcal{I}\mathcal{L}_1\\ |\bar b|=m}} \frac{\varphi_{\bar b}(y_s)}{m!}(\delta y_{ts})^{\bar b} + O\big(|t-s|^{n\kappa}\big) \\
&= \sum_{m=1}^{n-1}\sum_{\tau^1,\dots,\tau^m\in\mathcal{F}_{\mathcal{L}}^{n-1}}\sum_{\substack{\bar b\in\mathcal{I}\mathcal{L}_1\\ |\bar b|=m}} \frac{\varphi_{\bar b}(y_s)}{m!} y_s^{\tau^1,b_1}\cdots y_s^{\tau^m,b_m} X_{ts}^{\tau^1\cdots\tau^m} + O\big(|t-s|^{n\kappa}\big)
\end{aligned}
$$

which gives the required result. To show that every $z^{\tau}$ satisfies the $\delta$-equations (44) we can use a truncated version of the arguments used in Theorem 5.2. We omit the details. $\square$

The previous lemma shows that controlled paths are compatible with the application of non-linear functions. We will now prove that there exists an extension of the integral maps $\{I^a\}_a$ to the algebra $\mathcal{Q}_\gamma(X)$.

**Theorem 8.5.** *The integral maps $\{I^a\}_{a\in\mathcal{L}}$ can be extended to maps $I^a : \mathcal{Q}_\kappa(X) \to \delta\mathcal{Q}_\kappa(X)$. If $y \in \mathcal{Q}_\kappa(X)$ then $\delta z = I^a(y)$ is such that*

$$\delta z = X^{\bullet_a} z^{\bullet_a} + \sum_{\tau\in\mathcal{T}^n_{\mathcal{L}}} X^\tau z^\tau + z^\flat \tag{45}$$

*where $z^{\bullet_a} = y$, $z^{[\tau]_a} = y^\tau$ and zero otherwise. Moreover*

$$z^\flat = \Lambda\left[\sum_{\tau\in\mathcal{F}^{n-1}_{\mathcal{L}}\cup\{\emptyset\}} X^{B^+_a(\tau)} y^{\tau,\sharp}\right] \in \mathcal{C}_2^{\kappa(n+1)}.$$

**Proof.** Let $h = \sum_{\tau\in\mathcal{F}^{n-1}_{\mathcal{L}}} X^\tau y^\tau$ so that $\delta y = h + y^\sharp$. By linearity and by the definition of $X$ we have $h \in \mathcal{D}_I$ and

$$I^a(h) = \sum_{\tau\in\mathcal{F}^{n-1}_{\mathcal{L}}} I^a\bigl(X^\tau\bigr) y^\tau = \sum_{\tau\in\mathcal{F}^{n-1}_{\mathcal{L}}} X^{[\tau]_a} y^\tau = I^a\bigl(\delta y - y^\sharp\bigr).$$

We would like to show that we can extend $I^a$ such that $I^a(y^\sharp)$ is well defined so that we can set

$$I^a(\delta y) = \sum_{\tau\in\mathcal{F}^{n-1}_{\mathcal{L}}} X^{[\tau]_a} y^\tau + I^a\bigl(y^\sharp\bigr).$$

To do this we compute the action of $\delta$ on $I^a(y^\sharp)$. Since we want to preserve the properties of $I^a$ we have to require that

$$\delta I^a\bigl(y^\sharp\bigr) = I^a(e) y^\sharp + \sum_{\tau\in\mathcal{F}^{n-1}_{\mathcal{L}}} I^a\bigl(X^\tau\bigr) y^{\tau,\sharp} = X^{\bullet_a} y^\sharp + \sum_{\tau\in\mathcal{F}^{n-1}_{\mathcal{L}}} X^{[\tau]_a} y^{\tau,\sharp}$$

where we used the computation of $\delta y^{\tau,\sharp}$ in Lemma 8.3. Since $X$ is a $\gamma$-BRP and $y \in \mathcal{Q}_\kappa(X)$ with $1/(n+1) < \kappa < \gamma$ we see that the r.h.s. of this equation belongs to $\mathcal{Z}\mathcal{C}_3^{(n+1)\kappa} \subset \mathcal{Z}\mathcal{C}_3^{1+}$ so that it belongs to the domain of the $\Lambda$ map and then we can define

$$I^a\bigl(y^\sharp\bigr) = \Lambda\left[X^{\bullet_a} y^\sharp + \sum_{\tau\in\mathcal{F}^{n-1}_{\mathcal{L}}} X^{[\tau]_a} y^{\tau,\sharp}\right]$$

which proves out statement taking into account that we can set $z = I^a(y) = I^a(1)y + I^a(\delta y)$. $\square$

**Example 8.6.** Let us continue our one-dimensional example. For the integral $z = I(y)$ of the controlled path $y$ introduced in Example 8.2 we get

$$\begin{aligned}\delta z = \delta I(y) &= X^{\bullet} y + X^{[\bullet]} y^{\bullet} + X^{[[\bullet]]} y^{[\bullet]} + X^{[\bullet\bullet]} y^{\bullet\bullet} + X^{[\bullet[\bullet]]} y^{[\bullet]\bullet} + X^{[[\bullet\bullet]]} y^{[\bullet\bullet]} + X^{[\bullet\bullet\bullet]} y^{\bullet\bullet\bullet} + X^{[[[\bullet]]]} y^{[[\bullet]]} + z^{\flat} \\ &= X^{\bullet} z^{\bullet} + X^{[\bullet]} z^{[\bullet]} + X^{[[\bullet]]} z^{[[\bullet]]} + X^{[\bullet\bullet]} z^{[\bullet\bullet]} + z^{\sharp}\end{aligned}$$

with

$$z^{\flat} = \Lambda\big[X^{\bullet} y^{\sharp} + X^{[\bullet]} y^{\bullet,\sharp} + X^{[[\bullet]]} y^{[\bullet],\sharp} + X^{[\bullet\bullet]} y^{\bullet\bullet,\sharp} + X^{[[\bullet\bullet]]} y^{[\bullet\bullet],\sharp} + X^{[\bullet[\bullet]]} y^{[\bullet]\bullet,\sharp} + X^{[\bullet\bullet\bullet]} y^{\bullet\bullet\bullet,\sharp}\big],$$

and the coefficients satisfy:

$$\begin{aligned}\delta z^{\bullet} = \delta y &= X^{\bullet} y^{\bullet} + X^{[\bullet]} y^{[\bullet]} + X^{\bullet\bullet} y^{\bullet\bullet} + X^{[\bullet\bullet]} y^{[\bullet\bullet]} + X^{[\bullet]\bullet} y^{[\bullet]\bullet} + X^{[\bullet\bullet]} y^{[\bullet\bullet]} + X^{\bullet\bullet\bullet} y^{\bullet\bullet\bullet} + X^{[[\bullet]]} y^{[[\bullet]]} + y^{\sharp} \\ &= X^{\bullet} z^{[\bullet]} + X^{[\bullet]} y^{[[\bullet]]} + X^{\bullet\bullet} z^{[\bullet\bullet]} + z^{\bullet,\sharp}, \\ \delta z^{[\bullet]} = \delta y^{\bullet} &= X^{\bullet}\big(y^{[\bullet]} + 2y^{\bullet\bullet}\big) + X^{[\bullet]}\big(y^{[[\bullet]]} + y^{[\bullet]\bullet}\big) + X^{\bullet\bullet}\big(y^{[\bullet]\bullet} + y^{[\bullet\bullet]} + 3y^{\bullet\bullet\bullet}\big) + y^{\bullet,\sharp} \\ &= X^{\bullet}\big(z^{[[\bullet]]} + 2z^{[\bullet\bullet]}\big) + z^{[\bullet],\sharp}, \\ \delta z^{[[\bullet]]} = \delta y^{[\bullet]} &= X^{\bullet}\big(y^{[\bullet]\bullet} + 2y^{[\bullet\bullet]} + y^{[[\bullet]]}\big) + y^{[\bullet],\sharp} \\ &= z^{[[\bullet]],\sharp}, \\ \delta z^{[\bullet\bullet]} = \delta y^{\bullet\bullet} &= X^{\bullet}\big(y^{[\bullet]\bullet} + y^{\bullet\bullet\bullet}\big) + y^{\bullet\bullet,\sharp} \\ &= z^{[\bullet\bullet],\sharp}.\end{aligned}$$

**Remark 8.7.** Given a controlled path $y \in \mathcal{Q}_\kappa(X; \mathbb{R}^n \otimes \mathbb{R}^d)$ we can lift it to a branched rough path $Y$ indexed by $\mathcal{T}_{\mathcal{L}_1}$ by the following recursion

$$Y^{\bullet_b} = \sum_{a \in \mathcal{L}} I^a\big(y^{ab}\big), \qquad Y^{[\tau^1 \cdots \tau^k]_b} = \sum_{a \in \mathcal{L}} I^a\big(y^{ab} Y^{\tau^1} \circ \cdots \circ Y^{\tau_k}\big), \quad b \in \mathcal{L}_1.$$

Indeed $\{J^b(\cdot) = \sum_{a \in \mathcal{L}} I^a(y^{ab} \cdot)\}_{b \in \mathcal{L}_1}$ defines a family of integrals in the sense of Definition 7.1 and $Y$ is the associated $\gamma$-BRP.

## 8.1. Rough differential equations

Let $f_a \in C(\mathbb{R}^k; \mathbb{R}^k)$, $a = 1, \ldots, d$, a family of vectorfields on $\mathbb{R}^k$. Given a family on integral map $I^a$ which define a $\gamma$-BRP $X$ we consider the *rough differential equation*

$$\delta y = \sum_{a \in \mathcal{L}} I^a\big(f_a(y)\big), \qquad y_0 = \eta \in \mathbb{R}^k \tag{46}$$

in the time interval $[0, T]$. This equation has a well-defined meaning when the vectorfields $f_a$ are $C_b^n$ with $n$ the largest integer for which $n\gamma \leqslant 1$. In this case we can look for solutions of the above equation with $y \in \mathcal{Q}_\gamma(X; \mathbb{R}^k)$ and Eq. (46) can be understood as a fixed point problem in $\mathcal{Q}_\gamma(X; \mathbb{R}^k)$

since we have that the map $\Gamma$ defined as

$$\delta\Gamma(y) = \sum_{a\in\mathcal{L}} I^a\big(f_a(y)\big), \qquad \Gamma(y)_0 = \eta$$

is well defined from $\mathcal{Q}_\gamma(X;\mathbb{R}^k)$ onto itself thanks to Lemma 8.4 and Theorem 8.5.

**Theorem 8.8.** *If $\{f_a\}_{a\in\mathcal{L}}$ is a family of $C^n_b$ vectorfields then the rough differential equation* (46) *has a global solution $y \in \mathcal{Q}_\gamma(X;\mathbb{R}^k)$ for any initial condition $\eta \in \mathbb{R}^k$.*

*If the vectorfields are $C^{n+1}_b$ the solution $\Phi(\eta, X) \in \mathcal{Q}_\gamma(X;\mathbb{R}^k)$ is unique and the map $\Phi : \mathbb{R}^k \times \Omega^\gamma_{T,\mathcal{L}} \to \mathcal{Q}_\gamma(X;\mathbb{R}^k)$ is Lipschitz in any finite interval* $[0, T]$.

**Proof.** The proof of existence is based on a compactness argument on the map $\Gamma$. Global solutions are obtained exploiting the boundedness of the vectorfields (and of their derivatives). Uniqueness is proven by contraction on sufficiently small time interval $[0, S]$. The arguments are just direct adaptation of the proof of similar statements which can be found in [23] and are quite standard so we prefer to omit them. □

## 9. Infinite dimensional rough equations

Another motivation to introduce a rough path theory based on tree-indexed iterated integrals comes from the observation that infinite dimensional differential equations generate quite naturally expansions in trees which cannot be reduced to "linear" iterated integrals by the means of some geometric property. We still do not have a general theory of such equations but in this section we would like to justify our point of view by the means of three examples which we have studied in detail elsewhere [26,30,27]: the 1d periodic deterministic Korteweg–de Vries (KdV) equation, Navier–Stokes like equations and a class of stochastic partial differential equations. Given the illustrative purpose of this section we will keep the exposition at a formal level. Rigorous results can be found in the papers cited above.

### *9.1. The KdV equation*

The 1d periodic KdV equation is the partial differential equation

$$\partial_t u(t,\xi) + \partial^3_\xi u(t,\xi) + \frac{1}{2}\partial_\xi u(t,\xi)^2 = 0, \qquad u(0,\xi) = u_0(\xi), \quad (t,\xi) \in \mathbb{R}\times\mathbb{T}, \tag{47}$$

where the initial condition $u_0$ belongs to some Sobolev space $H^\alpha(\mathbb{T})$ of the torus $\mathbb{T} = [-\pi, \pi]$. This equation has many interesting features (e.g. it is a completely integrable system) but here we are interested only in the interplay between the non-linear term and the dispersive linear term which is the generator of the Airy group $U(t)$ of isometries of $H^\alpha$. By going to Fourier variables and setting $v_t = U(t)u_t$ we recast the above equation in integral form

$$v_t(k) = v_0(k) + \frac{ik}{2}\sum_{k_1}{}' \int_0^t e^{-i3kk_1k_2 s} v_s(k_1)v_s(k_2)\,ds, \quad t \in [0,T],\ k \in \mathcal{Z}_*, \tag{48}$$

where $k_2 = k - k_1$ and $v_0(k) = u_0(k)$ and where the primed summation excludes the values $k_1 = 0$ and $k_1 = k$. We restrict our attention to initial conditions such that $v_0(0) = 0$. By introducing the linear operator $\dot X_\sigma(\varphi,\varphi) = \frac{ik}{2}\sum'_{k_1} e^{-i3kk_1k_2\sigma}\varphi(k_1)\varphi(k_2)$ this equation takes the abstract form

$$v_t = v_s + \int_s^t \dot X_\sigma(v_\sigma, v_\sigma)\,d\sigma, \quad t,s\in[0,T].$$

By iteratively substituting the unknown in this integral equation we obtain an expansion whose first terms look like

$$\begin{aligned} v_t = v_s &+ \int_s^t d\sigma\,\dot X_\sigma(v_s,v_s) + 2\int_s^t d\sigma\,\dot X_\sigma\left(v_s, \int_s^\sigma d\sigma_1\,\dot X_{\sigma_1}(v_s,v_s)\right) \\ &+ \int_s^t d\sigma\,\dot X_\sigma\left(\int_s^\sigma d\sigma_1\,\dot X_{\sigma_1}(v_s,v_s), \int_s^\sigma d\sigma_2\,\dot X_{\sigma_2}(v_s,v_s)\right) \\ &+ 4\int_s^t d\sigma\,\dot X_\sigma\left(v_s, \int_s^\sigma d\sigma_1\,\dot X_{\sigma_1}\left(v_s, \int_s^{\sigma_1} d\sigma_2\,\dot X_{\sigma_2}(v_s,v_s)\right)\right) + r_{ts} \end{aligned} \tag{49}$$

where $r_{ts}$ stands for the remaining terms in the expansion. Denote with $\mathcal{T}_{BP}\subseteq\mathcal{T}$ the set of (unlabeled) planar rooted trees with at most two branches at each node. A planar tree is a rooted tree endowed with an ordering of the branches at each node. Then each of the terms in this expansion can be associated to a tree in $\mathcal{T}_{BP}$ and we can define recursively multilinear operators $X^\tau$ as

$$X^{\bullet}_{ts}(\varphi_1,\varphi_2) = \int_s^t \dot X_\sigma(\varphi_1,\varphi_2)\,d\sigma,$$

$$X^{[\tau^1]}_{ts}(\varphi_1,\dots,\varphi_{m+1}) = \int_s^t \dot X_\sigma\big(X^{\tau^1}_{\sigma s}(\varphi_1,\dots,\varphi_m),\varphi_{m+1}\big)\,d\sigma$$

and

$$X^{[\tau^1\tau^2]}_{ts}(\varphi_1,\dots,\varphi_{m+n}) = \int_s^t \dot X_\sigma\big(X^{\tau^1}_{\sigma s}(\varphi_1,\dots,\varphi_m), X^{\tau^2}_{\sigma s}(\varphi_{m+1},\dots,\varphi_{m+n})\big)\,d\sigma.$$

Eq. (49) has then the form

$$\delta v = X^{\bullet}\big(v^{\times 2}\big) + X^{[\bullet]}\big(v^{\times 3}\big) + X^{[[\bullet]]}\big(v^{\times 4}\big) + X^{[\bullet\bullet]}\big(v^{\times 4}\big) + r \tag{50}$$

as an equation for $k$-increments where $v_s^{\times n} = (v_s,\dots,v_s)$ ($n$ times). Moreover we have algebraic relations for the $X^\tau$-s, for example

$$\delta X^{[\bullet]}(\varphi_1,\varphi_2,\varphi_3) = X^{\bullet}\big(X^{\bullet}(\varphi_1,\varphi_2),\varphi_3\big),$$

$$\delta X^{[[\bullet]]}(\varphi_1,\varphi_2,\varphi_3,\varphi_4) = X^{\bullet}\big(X^{[\bullet]}(\varphi_1,\varphi_2,\varphi_3),\varphi_4\big) + X^{[\bullet]}\big(X^{\bullet}(\varphi_1,\varphi_2),\varphi_3,\varphi_4\big),$$

and

$$\delta X^{[\bullet\bullet]}(\varphi_1,\varphi_2,\varphi_3,\varphi_4) = X^{\bullet}\big(X^{\bullet}(\varphi_1,\varphi_2), X^{\bullet}(\varphi_3,\varphi_4)\big) \\ + X^{[\bullet]}\big(\varphi_1,\varphi_2, X^{\bullet}(\varphi_3,\varphi_4)\big) + X^{[\bullet]}\big(\varphi_3,\varphi_4, X^{\bullet}(\varphi_1,\varphi_2)\big)$$

where we used the symmetry of the operator $\dot{X}$ to obtain this last equation. These relations have much in common with the analogous relations for branched rough paths, however here the additional information of the position of the various arguments must be taken into account in the combinatorics of the reduced coproduct. It would be interesting to determine a Hopf algebra structure on $\mathcal{T}_{BP}$ which could account for these relations in a general way.

Our interest in the $X$-operators comes from the fact that they are, usually, more regular than the original operator $\dot{X}$. This additional regularity usually comes at the expense of their Hölder time regularity when considered as operator-valued increments. We are then naturally led to consider Eq. (50) as a rough equation and to try to solve it using the $\Lambda$ map. For example using only up to the double iterated integrals we would obtain the equation

$$\delta v = (1-\Lambda\delta)\big[X^{\bullet}\big(v^{\times 2}\big) + X^{[\bullet]}\big(v^{\times 3}\big)\big]$$

which in some cases can be solved by fixed point methods. This strategy has allowed us to obtain solutions of the KdV equation for initial data in $H^{\alpha}$ with any $\alpha > -1/2$.

### 9.2. Navier–Stokes-like equations

The $d$-dimensional NS equation (or the Burgers' equation) have the abstract form

$$u_t = S_t u_0 + \int_0^t S_{t-s} B(u_s, u_s)\, ds \tag{51}$$

where $S$ is a bounded semi-group on a Banach space $\mathcal{B}$ and $B$ is a symmetric bilinear operator which is usually defined only on a subspace of $\mathcal{B}$. Here we cannot proceed as in the previous section since $S$ is only a semi-group and we must cope with the convolution directly. In [30] we showed that the solutions of this equation in the case of the 3d NS equation have the series representation

$$u_t = S_t u_0 + \sum_{\tau \in \mathcal{T}_B} X^{\tau}_{t0}\big(u_0^{\times d(\tau)}\big) \tag{52}$$

where $d(\tau)$ is a degree function and the $d(\tau)$-multilinear operator $X^{\tau}$ has recursive definition

$$X^{\bullet}_{ts}\big(\varphi^{\times 2}\big) = \int_s^t S_{t-u} B(S_{u-s}\varphi, S_{u-s}\varphi)\, du,$$

$$X^{[\tau^1]}_{ts}\big(\varphi^{\times(d(\tau^1)+1)}\big) = \int_s^t S_{t-u} B\big(X^{\tau^1}_{us}\big(\varphi^{\times d(\tau^1)}\big), \varphi\big)\, du$$

and

$$X^{[\tau^1\tau^2]}_{ts}\big(\varphi^{\times(d(\tau^1)+d(\tau^2))}\big) = \int_s^t S_{t-u} B\big(X^{\tau^1}_{us}\big(\varphi^{\times d(\tau^1)}\big), X^{\tau^2}_{us}\big(\varphi^{\times d(\tau^2)}\big)\big)\, du.$$

These operators can be shown to allow bounds in $\mathcal{B}$ of the form

$$\left|X^\tau\left(\varphi^{\times d(\tau)}\right)\right|_{\mathcal{B}} \leqslant C\frac{|t-s|^{\varepsilon|\tau|}}{(\tau!)^\varepsilon}|\varphi|_{\mathcal{B}}^{d(\tau)}$$

where $\varepsilon \geqslant 0$ is a constant depending on the particular Banach space $\mathcal{B}$ we choose. The series (52) can be shown to be norm convergent at least for small $t$ and define local solution of NS. Due to the presence of the convolution integral these $X$ operators do not behave nicely with respect to $\delta$. In [27] we introduced cochain complex $(\hat{\mathcal{C}}_*, \tilde{\delta})$ adapted to the study of such convolution integrals where the coboundary is given by $\tilde{\delta}h = \delta h - ah - ha$ with $a_{ts} = S_{t-s} - \mathrm{Id}$ the 2-increment naturally associated to the semi-group. There exists also a corresponding $\tilde{\Lambda}$-map which provides an appropriate inverse to $\tilde{\delta}$. Algebraic relations for these iterated integrals have then by-now familiar expressions, e.g.:

$$\tilde{\delta}X^{\text{\textbullet}\!\!\!\mid\!\text{\textbullet}}\left(\varphi^{\times 3}\right) = X^{\bullet}\left(X^{\bullet}\left(\varphi^{\times 2}\right), \varphi\right)$$

and so on.

*9.3. Polynomial SPDEs*

In the paper [27] we study path-wise solutions to SPDEs in the mild form

$$u_t = S_t u_0 + \int_0^t S_{t-s}\, dw_s\, f(u_s) \tag{53}$$

where the solution $u_t$ lives in some Hilbert space $\mathcal{B}$, $S$ is an analytic semi-group in $\mathcal{B}$, $f : \mathcal{B} \to \mathcal{V}$ some non-linear function with values another Hilbert space $\mathcal{V}$ and $w$ a Gaussian stochastic process with values in the space of linear operators from $\mathcal{V}$ to $\mathcal{B}$ (possibly unbounded). Like in the NS case above this abstract equation allows an expansion in trees when the non-linear term is polynomial. For example taking $f(\varphi) = B(\varphi, \varphi)$ for some symmetric bilinear operator $B$ we get a stack of iterated integrals on the stochastic process $w$:

$$X_{ts}^{\bullet}\left(\varphi^{\times 2}\right) = \int_s^t S_{t-u}\, dw_u\, B(S_{u-s}\varphi, S_{u-s}\varphi),$$

$$X_{ts}^{[\tau^1]}\left(\varphi^{\times(d(\tau^1)+1)}\right) = \int_s^t S_{t-u}\, dw_u\, B\left(X_{us}^{\tau^1}\left(\varphi^{\times d(\tau^1)}\right), \varphi\right)$$

and

$$X_{ts}^{[\tau^1\tau^2]}\left(\varphi^{\times(d(\tau^1)+d(\tau^2))}\right) = \int_s^t S_{t-u}\, dw_u\, B\left(X_{us}^{\tau^1}\left(\varphi^{\times d(\tau^1)}\right), X_{us}^{\tau^2}\left(\varphi^{\times d(\tau^2)}\right)\right),$$

where these integrals can be defined by stochastic integration with respect to the process $w$ (Itô or Stratonovich). So, provided useful (path-wise) estimates for these operators are available, we can use the $(\hat{\mathcal{C}}, \tilde{\delta})$ complex and the $\tilde{\Lambda}$ map to set up rough equations and study path-wise solutions of polynomial SPDE like Eq. (53).

## Appendix A. A variant of Lyons' neo-classical inequality

**Proposition A.1.** *For any $\gamma \in (0,1]$ there exists a constant $c_\gamma$ such that*

$$\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma(n-k)}}{(k!)^\gamma ((n-k)!)^\gamma} \leqslant c_\gamma \frac{(a+b)^{\gamma n}}{(n!)^\gamma} \tag{A.1}$$

*for any $a, b > 0$ and $n \geqslant 0$.*

**Proof.** Using Stirling's asymptotic for the factorial: $n! = e^{n(\log n - 1)}\sqrt{2\pi n}(1 + O(1/n))$ as $n \to \infty$ we can bound the sum $S_n$ on the l.h.s. of Eq. (A.1) by

$$S_n \leqslant \frac{a^{\gamma n}}{(n!)^\gamma} + \frac{b^{\gamma n}}{(n!)^\gamma} + \sum_{k=1}^{n-1} a^{\gamma k} b^{\gamma(n-k)} \frac{e^{\gamma k(1-\log k)+\gamma(n-k)(1-\log(n-k))+d}}{(2\pi)^\gamma k^\gamma (n-k)^\gamma} g(k)$$

where $g \geqslant 1$ is a bounded function such that $g(k) \to 1$ as $k \to \infty$ and $n - k \to \infty$. Let $\varphi(x) = x\log(x/a) + (1-x)\log[(1-x)/b] + \log(a+b)$, then

$$(n!)^\gamma (a+b)^{-\gamma n} S_n \leqslant \left(\frac{a}{a+b}\right)^{\gamma n} + \left(\frac{b}{a+b}\right)^{\gamma n} + \sum_{k=1}^{n-1} (n!)^\gamma \frac{e^{\gamma(n-\log n)-\gamma n\varphi(k/n)}}{(2\pi)^\gamma k^\gamma (n-k)^\gamma}.$$

Using again the asymptotic formula for $n!$ we get

$$(n!)^\gamma (a+b)^{-\gamma n} S_n \leqslant 2 + \sum_{k=1}^{n-1} \frac{n^\gamma e^{-\gamma n\varphi(k/n)}}{(2\pi)^{\gamma/2} k^\gamma (n-k)^\gamma} g'(k), \tag{A.2}$$

where $g'$ is another function with the same properties as $g$. The function $\varphi$ has minimum in $a/(a+b)$ and $\varphi(a/(a+b)) = 0$. In the limit $n \to \infty$ the contributions to the sum coming from the terms for which $|k/n - a/(a+b)| > \varepsilon$ is exponentially suppressed. Moreover $\varphi''(a/(a+b)) = (a+b)^2/(ab) \geqslant 1$ so the sum for the values of $k$ for which $|k/n - a/(a+b)| \leqslant \varepsilon$ can be bounded by a Gaussian integral uniformly in $a, b$. Then the r.h.s. of Eq. (A.2) can be bounded by a constant independent of $a, b$. $\square$

**Remark A.2.** The same approach can be used to prove the original neo-classical inequality if we do not care for optimality of the constant.

## References


[1] J.C. Butcher, Numerical Methods for Ordinary Differential Equations, John Wiley & Sons Ltd., Chichester, 2003.
[2] J.C. Butcher, An algebraic theory of integration methods, Math. Comp. 26 (1972) 79–106.
[3] P. Cayley, On the analytical forms called trees, Amer. J. Math. 4 (1/4) (1881) 266–268.
[4] E. Hairer, G. Wanner, On the Butcher group and general multi-value methods, Computing (Arch. Elektron. Rechnen) 13 (1) (1974) 1–15.
[5] E. Hairer, S.P. Nørsett, G. Wanner, Solving Ordinary Differential Equations, I. Nonstiff Problems, second ed., Springer Ser. Comput. Math., vol. 8, Springer-Verlag, Berlin, 1993.
[6] A. Connes, D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Comm. Math. Phys. 199 (1) (1998) 203–242.
[7] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem, I. The Hopf algebra structure of graphs and the main theorem, Comm. Math. Phys. 210 (1) (2000) 249–273.
[8] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem, II. The $\beta$-function, diffeomorphisms and the renormalization group, Comm. Math. Phys. 216 (1) (2001) 215–241.
[9] C. Brouder, Trees, renormalization and differential equations, BIT 44 (3) (2004) 425–438.
[10] C. Brouder, Runge–Kutta methods and renormalization, Eur. Phys. J. C 12 (2000) 521–534.

[11] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories, Adv. Theor. Math. Phys. 2 (2) (1998) 303–334.
[12] D. Kreimer, Chen's iterated integral represents the operator product expansion, Adv. Theor. Math. Phys. 3 (3) (1999) 627–670.
[13] A. Dür, Möbius Functions, Incidence Algebras and Power Series Representations, Lecture Notes in Math., vol. 1202, Springer-Verlag, Berlin, 1986.
[14] M.E. Sweedler, Hopf Algebras, Math. Lecture Note Ser., W.A. Benjamin, Inc., New York, 1969.
[15] M.E. Hoffman, Combinatorics of rooted trees and Hopf algebras, Trans. Amer. Math. Soc. 355 (9) (2003) 3795–3811 (electronic).
[16] L. Foissy, Les algèbres de Hopf des arbres enracinés décorés, II, Bull. Sci. Math. 126 (4) (2002) 249–288.
[17] L. Foissy, Les algèbres de Hopf des arbres enracinés décorés, I, Bull. Sci. Math. 126 (3) (2002) 193–239.
[18] K.T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (5) (1977) 831–879.
[19] K.-T. Chen, Collected papers of K.-T. Chen, Contemporary Mathematicians, Birkhäuser Boston Inc., Boston, MA, 2001, edited and with a preface by Philippe Tondeur, and an essay on Chen's life and work by Richard Hain and Tondeur.
[20] T.J. Lyons, Differential equations driven by rough signals, Rev. Mat. Iberoamericana 14 (2) (1998) 215–310.
[21] T. Lyons, Z. Qian, System Control and Rough Paths, Oxford Math. Monogr., Oxford University Press, Oxford, 2002, Oxford Science Publications.
[22] A. Lejay, An introduction to rough paths, in: Séminaire de Probabilités XXXVII, in: Lecture Notes in Math., vol. 1832, Springer, Berlin, 2003, pp. 1–59.
[23] M. Gubinelli, Controlling rough paths, J. Funct. Anal. 216 (1) (2004) 86–140.
[24] D. Feyel, A. de La Pradelle, Curvilinear integrals along enriched paths, Electron. J. Probab. 11 (2006) 860–892.
[25] P. Friz, N. Victoir, Approximations of the Brownian rough path with applications to stochastic analysis, Ann. Inst. H. Poincaré Probab. Statist. 41 (4) (2005) 703–724.
[26] M. Gubinelli, Rough solutions of the periodic Korteweg–de Vries equation, preprint, 2006.
[27] M. Gubinelli, S. Tindel, Rough evolution equations, Ann. Probab. (2010), in press.
[28] A. Neuenkirch, I. Nourdin, A. Rößler, S. Tindel, Trees and asymptotic developments for fractional stochastic differential equations, Ann. Inst. H. Poincaré Probab. Statist. 45 (1) (2009) 157–174.
[29] A. Rößler, Rooted tree analysis for order conditions of stochastic Runge–Kutta methods for the weak approximation of stochastic differential equations, Stoch. Anal. Appl. 24 (1) (2006) 97–134.
[30] M. Gubinelli, Rooted trees for 3D Navier–Stokes equation, Dyn. Partial Differ. Equ. 3 (2) (2006) 161–172.